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Allocation in stratified sampling based on
preliminary tests of significance

by

Victor Kuang-Tao Tang

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Head of Major Department

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I. INTRODUCTION

Sampling techniques are being increasingly used by business, government and industry to provide data for decision making, policy formulation, planning and developmental activities in almost all fields covering human activity. It is, therefore, all the more important that proper care is exercised in the designing of sample surveys as also in the subsequent analysis of the data collected therefrom. The precision of the results obtained from a sample survey depends not only on the size of the sample but also on various other aspects of the sample design such as the manner in which the sample is selected, the manner in which the estimates are formed, etc. To have an efficient survey design, it is therefore necessary to make effective use of all the available resources including physical facilities such as field staff, equipment, etc. and information about the population to be sampled.

One of the designs used frequently in sample surveys is stratified sampling. The precision of the estimated population mean depends mainly on the sample sizes n_i to be drawn from the different strata which can be fixed at statisticians' will. In the classical sense, the n_i are so chosen as to maximize the precision for given cost or minimize cost for given degree of precision. It is well known that if the cost per unit is the same from stratum to stratum and n_i the sample

size to be drawn from the i th stratum is proportional to $N_i \sigma_i$ where N_i is the number of units in the i th stratum and σ_i^2 is the variance for the i th stratum, then the population total or mean can be estimated with maximum precision for a given cost. This is often referred to as Neyman allocation. For example, see Solomon and Zacks (1970). The problem of optimum allocation has been considered by several authors. Ericson (1965, 1968) has considered the problem from a Bayesian point of view when prior information concerning the strata means is available. Zacks (1970) and Grosh (1969) have discussed the case when the observed variable takes only two values - zero or one.

Neyman allocation depends upon strata variances σ_i^2 which are generally not known. One way to overcome this difficulty is to use the technique of two-phase sampling introduced by Sukhatme (1935). The technique consists in first drawing a preliminary sample of fixed size from each stratum to estimate σ_i^2 which in turn are used to estimate the optimum sample sizes n_i to be drawn from the different strata. This allocation will be called modified Neyman allocation. The problem has also been considered by Draper and Guttman (1968a, 1968b) from a Bayesian point of view. They have obtained some results concerning the optimum allocation among the different strata at the second phase using information obtained from the first phase.

Another allocation which is commonly used in practice and does not require knowledge of the population variances σ_i^2 is proportional allocation. If the strata variances σ_i^2 differ significantly among themselves, Evans (1951) has obtained conditions under which modified Neyman allocation is more efficient than proportional allocation. If, however, the strata variances σ_i^2 do not differ significantly among themselves, the modified Neyman allocation may turn out to be less efficient than proportional allocation. Before deciding on the method of allocation, it is therefore proposed to carry out a preliminary test of significance concerning the homogeneity of strata variances. If, on the basis of the test of significance, the strata variances are found to be homogeneous, the sample sizes to be drawn from the different strata will be determined according to proportional allocation. Otherwise, the sample sizes will be determined according to modified Neyman allocation.

The problem of estimation subsequent to tests of significance has been considered by several authors. For an excellent bibliography on the subject, reference may be made to Kitagawa (1963) and Bancroft (1964). A survey of the work done in this area shows that not much has been done in the field of sampling except for some work by Ruhl and Sedransk (1967). More recently, Carrillo (1969) considered the problem of estimation of variance in stratified sampling

subsequent to preliminary test of significance of the homogeneity of strata variances. We shall consider in detail the problem of allocation of sample sizes to the different strata based on preliminary test of significance and investigate its efficiency with respect to proportional allocation and modified Neyman allocation.

II. ALLOCATION OF SAMPLE SIZES TO DIFFERENT STRATA

Consider a population of N units which are classified into k strata, the i th stratum containing N_i units so that $\sum_{i=1}^k N_i = N$. Let Y be the characteristic under study and consider the problem of estimating the population mean $\bar{y}_N = \frac{1}{N} \sum_{i=1}^N y_i$ or the population total $N\bar{y}_N$ from a stratified random sample of size $n = \sum_{i=1}^k n_i$ where n_i units are drawn by simple random sampling without replacement from the i th stratum. An unbiased estimate of the population mean \bar{y}_N is given by

$$\bar{y}_w = \sum_{i=1}^k w_i \bar{y}_{n_i}, \quad (2.1)$$

where $w_i = N_i/N$ is the proportion of units in the i th stratum and \bar{y}_{n_i} is the simple mean estimate of \bar{y}_{N_i} , the mean of the i th stratum. The variance of the estimate \bar{y}_w is given by

$$V(\bar{y}_w) = \sum_{i=1}^k w_i^2 \left(\frac{1}{n_i} - \frac{1}{N_i} \right) S_i^2, \quad (2.2)$$

If N_i is so large that $\frac{N_i}{N_i - 1} \approx 1$, $V(\bar{y}_w)$ can be written as

$$V(\bar{y}_w) = \sum_{i=1}^k \frac{w_i^2 \sigma_i^2}{n_i} - \frac{1}{N} \sum_{i=1}^k w_i \sigma_i^2, \quad (2.3)$$

Suppose that C_0 is the total budget available for the survey and that c_i is the cost per unit of sampling in the i th

stratum. The classical problem of allocation of sample sizes in stratified sampling is to determine a vector $n=(n_1, \dots, n_k)$ of k non-negative integers that satisfy

$$\sum_{i=1}^k c_i n_i \leq C_0, \quad (2.4)$$

and for which $V(\bar{y}_w)$ is minimum. The optimum allocation so determined is given by

$$n_i = C_0 w_i \sigma_i / \sqrt{c_i} \sum_{i=1}^k w_i \sigma_i \sqrt{c_i}, \quad (2.5)$$

If $c_i=c$ for every i , the optimum allocation reduces to

$$n_i = n w_i \sigma_i / \sum_{i=1}^k w_i \sigma_i, \quad (2.6)$$

where $n=C_0/c$. This allocation is known as Neyman allocation.

Both the optimum allocation and Neyman allocation depend on strata variances σ_i^2 which are generally not known. A sample of fixed size m is therefore drawn from each stratum and used to estimate σ_i^2 which in turn are used to estimate n_i from 2.6. In this case, n_i is given by

$$n_i = n w_i s_i / \sum_{i=1}^k w_i s_i, \quad (2.7)$$

which we shall call modified Neyman allocation. Under this allocation, the conditional variance of the estimate \bar{y}_w is given by

$$V(\bar{y}_w | s_i^2)_N = \frac{1}{n} \sum_{i=1}^k w_i^2 S_i^2 - \frac{1}{N} \sum_{i=1}^k w_i^2 S_i^2 + \frac{1}{n} \sum_{i \neq j}^k w_i w_j S_i^2 \frac{s_j}{s_i}, \quad (2.8)$$

while the unconditional variance is given by

$$V(\bar{y}_w)_{\underline{N}} = \frac{1}{n} \sum_{i=1}^k w_i^2 S_i^2 - \frac{1}{N} \sum_{i=1}^k w_i S_i^2 + \frac{1}{n} \sum_{i \neq j}^k w_i w_j S_i^2 E(s_j/s_i) , \quad (2.9)$$

where the symbol \underline{N} stands for modified Neyman allocation. If N_i is so large that $\frac{N_i}{N_i-1} \approx 1$, then 2.9 can be written as

$$V(\bar{y}_w)_{\underline{N}} = \frac{1}{n} \sum_{i=1}^k w_i^2 \sigma_i^2 - \frac{1}{N} \sum_{i=1}^k w_i \sigma_i^2 + \frac{1}{n} \sum_{i \neq j}^k w_i w_j \sigma_i^2 E(s_j/s_i) . \quad (2.10)$$

If the sample sizes to the different strata are allocated according to proportional allocation, then

$$n_i = n w_i , \quad (2.11)$$

and the corresponding variance of the estimate \bar{y}_w is given by

$$V(\bar{y}_w)_P = \left(\frac{1}{n} - \frac{1}{N} \right) \sum_{i=1}^k w_i \sigma_i^2 , \quad (2.12)$$

where the symbol P stands for proportional allocation.

Finally, we shall consider sometimes proportional and sometimes modified Neyman allocation based on a preliminary test of significance. For short, we shall call it sometimes proportional allocation. Under this allocation

$$\begin{aligned} n_i &= n w_i && \text{if } s_j^2/s_i^2 < \lambda \quad \text{for all pairs } i \neq j \\ &= n w_i s_i / \sum_{i=1}^k w_i s_i && \text{otherwise,} \end{aligned} \quad (2.13)$$

where λ is a fixed constant.

To discuss the efficiency of sometimes proportional allocation with respect to proportional allocation and

modified Neyman allocation, it is necessary to evaluate the variance of the estimate \bar{y}_w under sometimes proportional allocation which is done in the next section.

III. VARIANCE OF \bar{y}_w UNDER SOMETIMES

PROPORTIONAL ALLOCATION

The evaluation of the variance of \bar{y}_w under sometimes proportional allocation is by no means straight forward and involves complicated algebra. We shall therefore first consider the relatively simple case of two strata when $k=2$ and then extend the results further.

A. Evaluation of $V(\bar{y}_w)_S$ for $k=2$ and $\sigma_1^2 \leq \sigma_2^2$

The sometimes proportional allocation now takes the form

$$\begin{aligned} n_i &= nw_i & \text{if } s_2^2/s_1^2 < \lambda \\ &= nw_i s_i / (w_1 s_1 + w_2 s_2) & \text{if } s_2^2/s_1^2 \geq \lambda, \end{aligned} \quad (3.1.1)$$

Let the event A_0 and its complementary event A_1 be defined by

$$A_0 : \{s_2^2/s_1^2 < \lambda\}$$

and

$$A_1 : \{s_2^2/s_1^2 \geq \lambda\} \quad . \quad (3.1.2)$$

The conditional variance of the estimate \bar{y}_w is given by

$$V(\bar{y}_w | s_i)_S = \sum_{i=0}^1 V(\bar{y}_w | A_i) P(A_i) ,$$

where the symbol S stands for sometimes proportional allocation. The unconditional variance of \bar{y}_w , denoted by $V(\bar{y}_w)_S$, is given by

$$V(\bar{y}_w)_S = \sum_{i=0}^1 E_i [V(\bar{y}_w | A_i)] P(A_i) , \quad (3.1.3)$$

where E_i denotes that the expectation is taken with reference to the set A_i , $i=0,1$.

Clearly,

$$E_0[V(\bar{y}_w|A_0)] = V(\bar{y}_w)_P = \left(\frac{1}{n} - \frac{1}{N}\right)\sigma_1^2(w_1+w_2\theta_{21}), \quad (3.1.4)$$

$$\text{where } \theta_{ji} = \sigma_j^2/\sigma_i^2, \quad (3.1.5)$$

$$\text{while } E_1[V(\bar{y}_w|A_1)] = \frac{1}{n} \sum_{i=1}^2 w_i^2 \sigma_i^2 - \frac{1}{N} \sum_{i=1}^2 w_i \sigma_i^2 + \frac{1}{n} \sum_{i \neq j}^2 w_i w_j \sigma_i^2 E(s_j/s_i|A_1). \quad (3.1.6)$$

To evaluate $E(s_j/s_i|A_i)$, we use a result due to Carrillo (1969) given below.

Lemma 3.1 Let s_1^2 and s_2^2 be independent unbiased estimates of σ_1^2 and σ_2^2 based on f_1 and f_2 degrees of freedom respectively. Then

$$\begin{aligned} E(s_1^{2t_1} s_2^{2t_2} | A_0) P(A_0) \\ = I_{q_{21}}(\tfrac{1}{2}f_2+t_2, \tfrac{1}{2}f_1+t_1) \prod_{i=1}^2 (2\sigma_i^2/f_i)^{t_i} \Gamma(\tfrac{1}{2}f_i+t_i) / \Gamma(\tfrac{1}{2}f_i), \end{aligned} \quad (3.1.7)$$

and

$$\begin{aligned} E(s_1^{2t_1} s_2^{2t_2} | A_1) P(A_1) \\ = I_{p_{21}}(\tfrac{1}{2}f_1+t_1, \tfrac{1}{2}f_2+t_2) \prod_{i=1}^2 (2\sigma_i^2/f_i)^{t_i} \Gamma(\tfrac{1}{2}f_i+t_i) / \Gamma(\tfrac{1}{2}f_i), \end{aligned} \quad (3.1.8)$$

where $I(..)$ is the incomplete beta distribution.

$$p_{21} = \frac{1}{1+(f_2\sigma_1^2/f_1\sigma_2^2)} \quad \text{and} \quad q_{21} = 1-p_{21}.$$

In what follows, we shall assume that the distribution

of the characteristic under study in the i th stratum can be approximated by normal distribution with unknown mean μ_i and unknown variance σ_i^2 , $i=1,2$. Letting $t_1 = -\frac{1}{2}$, $t_2 = \frac{1}{2}$, $f_1 = f_2 = f$ and $\lambda_{21} = \lambda$ in 3.1.8, we obtain

$$E(s_2/s_1 | A_1) P(A_1) = \frac{1}{2} G \theta_{21}^{\frac{1}{2}} I_{p_{21}} \left(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2} \right), \quad (3.1.9)$$

$$\text{where } G = 2 \Gamma(\frac{1}{2}f - \frac{1}{2}) \Gamma(\frac{1}{2}f + \frac{1}{2}) / \Gamma(\frac{1}{2}f) \Gamma(\frac{1}{2}f), \quad (3.1.10)$$

$$\text{and } p_{ji} = \theta_{ji} / (\theta_{ji} + \lambda). \quad (3.1.11)$$

$E(s_1/s_2 | A_1) P(A_1)$ can be obtained by symmetry, i.e.,

$$E(s_1/s_2 | A_1) P(A_1) = \frac{1}{2} G \theta_{21}^{-\frac{1}{2}} I_{p_{21}} \left(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2} \right). \quad (3.1.12)$$

To evaluate $P(A_0)$ let $t_1 = t_2 = 0$ in 3.1.7 and we get

$$P(A_0) = I_{q_{21}} \left(\frac{1}{2}f, \frac{1}{2}f \right), \quad (3.1.13)$$

$$\text{where } q_{ji} = 1 - p_{ji} = \lambda / (\theta_{ji} + \lambda). \quad (3.1.14)$$

$$\text{Hence } P(A_1) = 1 - P(A_0) = I_{p_{21}} \left(\frac{1}{2}f, \frac{1}{2}f \right). \quad (3.1.15)$$

Using 3.1.9, 3.1.12 and 3.1.15, we obtain

$$\begin{aligned} E_1 [V(\bar{y}_w | A_1)] &= \frac{1}{n} \sum_{i=1}^2 w_i^2 \sigma_i^2 - \frac{1}{N} \sum_{i=1}^2 w_i \sigma_i^2 + \frac{1}{2} G \frac{w_1 w_2}{n} \sigma_1^2 \theta_{21}^{\frac{1}{2}} [I_{p_{21}} \left(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2} \right) \\ &\quad + I_{p_{21}} \left(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2} \right)] / I_{p_{21}} \left(\frac{1}{2}f, \frac{1}{2}f \right). \end{aligned} \quad (3.1.16)$$

Substituting from 3.1.4, 3.1.13, 3.1.15 and 3.1.16 in 3.1.3, we obtain, after simplification,

$$V(\bar{y}_w)_S = \frac{\sigma_1^2}{n}(w_1^2 + w_2^2 \theta_{21}) - \frac{\sigma_1^2}{N}(w_1 + w_2 \theta_{21}) \\ + \frac{w_1 w_2}{n} \sigma_1^2 [(1 + \theta_{21}) I_{q_{21}}(\frac{1}{2}f, \frac{1}{2}f) + G \theta_{21}^{\frac{1}{2}} I_{p_{21}}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f - \frac{1}{2})] . \quad (3.1.17)$$

If we let λ tend to infinity, we obtain the variance of the estimate \bar{y}_w under proportional allocation, i.e.,

$$V(\bar{y}_w)_P = (\frac{1}{n} - \frac{1}{N}) \sigma_1^2 (w_1 + w_2 \theta_{21}) . \quad (3.1.18)$$

Putting $\lambda=0$, we obtain the variance of the estimate \bar{y}_w under modified Neyman allocation, i.e.,

$$V(\bar{y}_w)_N = \frac{\sigma_1^2}{n}(w_1^2 + w_2^2 \theta_{21}) - \frac{\sigma_1^2}{N}(w_1 + w_2 \theta_{21}) + \frac{w_1 w_2}{n} \sigma_1^2 G \theta_{21}^{\frac{1}{2}} . \quad (3.1.19)$$

B. Evaluation of $V(\bar{y}_w)_S$ for $k=2$ and $\sigma_1^2 \neq \sigma_2^2$

In this case, the sometimes proportional allocation takes the form

$$n_i = n w_i \quad \text{if } s_j^2 / s_i^2 < \lambda \quad \text{for } i, j=1, 2 \\ \text{and } i \neq j , \\ = n w_i s_i / (w_1 s_1 + w_2 s_2) \quad \text{otherwise.} \quad (3.2.1)$$

Let the event $A_0^!$ be defined by

$$A_0^! : \{s_j^2 / s_i^2 < \lambda \quad \text{for } i, j=1, 2 \text{ and } i \neq j\} , \quad (3.2.2)$$

and $A_1^!$, the complementary event of $A_0^!$.

Following the same procedure as in previous section, the variance of the estimate \bar{y}_w under sometimes proportional allocation is given by

$$V(\bar{y}_w)_S = \sum_{i=0}^1 E_i[V(\bar{y}_w|A_i^!)] P(A_i^!) , \quad (3.2.3)$$

where E_i denotes that the expectation is taken with reference to the set $A_i^!$, $i=0,1$.

Again,

$$E_0[V(\bar{y}_w|A_0^!)] = V(\bar{y}_w)_P = \left(\frac{1}{n} - \frac{1}{N}\right) \sigma_1^2 (w_1 + w_2 \theta_{21}) , \quad (3.2.4)$$

while

$$E_1[V(\bar{y}_w|A_1^!)] = \frac{1}{n} \sum_{i=1}^2 w_i^2 \sigma_i^2 - \frac{1}{N} \sum_{i=1}^2 w_i \sigma_i^2 + \frac{1}{n} \sum_{i \neq j}^2 w_i w_j \sigma_i^2 E(s_j/s_i|A_1^!) . \quad (3.2.5)$$

To evaluate $E(s_j/s_i|A_1^!)$, we use another result due to Carrillo (1969) given below.

Lemma 3.2 Let s_1^2 and s_2^2 be independent unbiased estimates of σ_1^2 and σ_2^2 based on f_1 and f_2 degrees of freedom respectively. Then

$$\begin{aligned} E(s_1^{2t_1} s_2^{2t_2} | A_0^!) P(A_0^!) \\ = [I_{p_{21}}^{(1)}(\tfrac{1}{2}f_1+t_1, \tfrac{1}{2}f_2+t_2) \\ - I_{p_{21}}^{(2)}(\tfrac{1}{2}f_1+t_1, \tfrac{1}{2}f_2+t_2)] \prod_{i=1}^2 (2\sigma_i^2/f_i)^{t_i} \Gamma(\tfrac{1}{2}f_i+t_i) / \Gamma(\tfrac{1}{2}f_i), \end{aligned} \quad (3.2.6)$$

and

$$\begin{aligned} E(s_1^{2t_1} s_2^{2t_2} | A_1^!) P(A_1^!) \\ = [I_{q_{21}}^{(1)}(\tfrac{1}{2}f_2+t_2, \tfrac{1}{2}f_1+t_1) \\ + I_{p_{21}}^{(2)}(\tfrac{1}{2}f_1+t_1, \tfrac{1}{2}f_2+t_2)] \prod_{i=1}^2 (2\sigma_i^2/f_i)^{t_i} \Gamma(\tfrac{1}{2}f_i+t_i) / \Gamma(\tfrac{1}{2}f_i), \end{aligned} \quad (3.2.7)$$

where $p_{21}^{(i)} = \frac{1}{1+(f_2\sigma_1^2\lambda_{21}^{(i)}/f_1\sigma_2^2)}$, $i=1,2$, and $q_{21}^{(1)}=1-p_{21}^{(1)}$.

Letting $t_1=-\frac{1}{2}$, $t_2=\frac{1}{2}$, $f_1=f_2=f$, $\lambda_{21}^{(1)}=1/\lambda$ and $\lambda_{21}^{(2)}=\lambda$ in

3.2.7, we obtain

$$E(s_2/s_1 | A_1') P(A_1') = \frac{1}{2} G \theta_{21}^{\frac{1}{2}} \left[I_{q_{21}}^{(1)} \left(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2} \right) + I_{p_{21}} \left(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2} \right) \right], \quad (3.2.8)$$

$$\text{where } q_{21}^{(1)} = 1 - p_{21}^{(1)} = 1/(1 + \lambda \theta_{21}), \quad (3.2.9)$$

and similarly,

$$E(s_1/s_2 | A_1') P(A_1') = \frac{1}{2} G \theta_{21}^{-\frac{1}{2}} \left[I_{q_{21}}^{(1)} \left(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2} \right) + I_{p_{21}} \left(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2} \right) \right]. \quad (3.2.10)$$

To evaluate $P(A_0')$, let $t_1=t_2=0$ in 3.2.6 and we get

$$P(A_0') = I_{p_{21}}^{(1)} \left(\frac{1}{2}f, \frac{1}{2}f \right) - I_{p_{21}} \left(\frac{1}{2}f, \frac{1}{2}f \right). \quad (3.2.11)$$

Hence

$$P(A_1') = 1 - P(A_0') = I_{q_{21}}^{(1)} \left(\frac{1}{2}f, \frac{1}{2}f \right) + I_{p_{21}} \left(\frac{1}{2}f, \frac{1}{2}f \right). \quad (3.2.12)$$

Using 3.2.8, 3.2.10 and 3.2.12, we obtain

$$\begin{aligned} E_1[V(\bar{y}_w | A_1)] &= \sigma_1^2(w_1^2 + w_2^2 \theta_{21})/n - \sigma_1^2(w_1 + w_2 \theta_{21})/N \\ &\quad + \frac{1}{2} G \frac{w_1 w_2}{n} \sigma_1^2 \theta_{21}^{\frac{1}{2}} \left[I_{q_{21}}^{(1)} \left(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2} \right) + I_{p_{21}} \left(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2} \right) \right. \\ &\quad \left. + I_{q_{21}}^{(1)} \left(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2} \right) + I_{p_{21}} \left(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2} \right) \right] / \left[I_{q_{21}}^{(1)} \left(\frac{1}{2}f, \frac{1}{2}f \right) \right. \\ &\quad \left. + I_{p_{21}} \left(\frac{1}{2}f, \frac{1}{2}f \right) \right]. \quad (3.2.13) \end{aligned}$$

Substituting from 3.2.4, 3.2.11, 3.2.12 and 3.2.13 in

3.2.3, we obtain, after simplification,

$$\begin{aligned}
 V(\bar{y}_w)_S &= \sigma_1^2(w_1^2 + w_2^2 \theta_{21})/n - \sigma_1^2(w_1 + w_2 \theta_{21})/N \\
 &+ \frac{w_1 w_2}{n} \sigma_1^2 \{ (1 + \theta_{21}) [I_{p_{21}}^{(1)}(\tfrac{1}{2}f, \tfrac{1}{2}f) - I_{p_{21}}(\tfrac{1}{2}f, \tfrac{1}{2}f)] \\
 &+ G\theta_{21}^{\frac{1}{2}} [I_{q_{21}}^{(1)}(\tfrac{1}{2}f - \tfrac{1}{2}, \tfrac{1}{2}f - \tfrac{1}{2}) + I_{p_{21}}(\tfrac{1}{2}f - \tfrac{1}{2}, \tfrac{1}{2}f - \tfrac{1}{2})] \}. \quad (3.2.14)
 \end{aligned}$$

If we let λ tend to infinity, we obtain the variance of the estimator \bar{y}_w under proportional allocation. Putting $\lambda=1$, we obtain the variance of \bar{y}_w under modified Neyman allocation. The expressions are as given in 3.1.18 and 3.1.19 respectively.

C. Evaluation of $V(\bar{y}_w)_S$ for $k=3$ and $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2$

In this case the sometimes proportional allocation takes the form

$$\begin{aligned}
 n_i &= nw_i && \text{if } s_j^2/s_i^2 < \lambda \text{ for } i, j=1, 2, 3 \\
 &&& \text{and } i < j, \\
 &= nw_i s_i / \sum_{i=1}^3 w_i s_i && \text{otherwise.}
 \end{aligned} \quad (3.3.1)$$

Let the event A_0'' be defined by

$$A_0'' : \{s_j^2/s_i^2 < \lambda \text{ for } i, j=1, 2, 3 \text{ and } i < j\}, \quad (3.3.2)$$

and let A_1'' be the complementary event of A_0'' . Then the variance of the estimate \bar{y}_w under sometimes proportional allocation is given by

$$V(\bar{y}_w)_S = \sum_{i=0}^1 E_i[V(\bar{y}_w|A_i'')] P(A_i''), \quad (3.3.3)$$

where E_i denotes that the expectation is taken with reference to the set A_i'' , $i=0,1$.

As before,

$$E_0[V(\bar{y}_w | A_0'')] = V(\bar{y}_w)_P = \left(\frac{1}{n} - \frac{1}{N}\right) \sum_{i=1}^3 w_i \sigma_i^2, \quad (3.3.4)$$

and

$$E_1[V(\bar{y}_w | A_1'')] = \frac{1}{n} \sum_{i=1}^3 w_i \sigma_i^2 - \frac{1}{N} \sum_{i=1}^3 w_i \sigma_i^2 + \frac{1}{n} \sum_{i=1}^3 \sum_{j \neq i} w_i w_j \sigma_i^2 E(s_j/s_i | A_1''). \quad (3.3.5)$$

To evaluate expectations of the type $E(s_j/s_i | A_1'')$, we shall first prove Lemma 3.5 given below.

Lemma 3.5 Let s_1^2 , s_2^2 and s_3^2 be independent

unbiased estimates of σ_1^2 , σ_2^2 and σ_3^2 respectively, all based on f degrees of freedom, where f is an even integer. Then if

i) t_1 is a non-negative integer

$$\begin{aligned} & E(s_1^{2t_1} s_2^{2t_2} s_3^{2t_3} | A_0'') P(A_0'') \\ &= K_3 \left\{ \sum_{r=0}^{\frac{1}{2}f+t_1-1} \frac{\Gamma(f+t_1+t_3-1-r)}{\Gamma(\frac{1}{2}f+t_3)\Gamma(\frac{1}{2}f+t_1-r)} q_{31}^{\frac{1}{2}f+t_3} p_{31}^{\frac{1}{2}f+t_1-1-r} \right. \\ & \quad \left[I_{q_{21}}^{\frac{1}{2}f+t_2, f+t_1+t_3-1-r} - I_{q_{21}^*}^{\frac{1}{2}f+t_2, f+t_1+t_3-1-r} \right] \\ & \quad + \sum_{r=0}^{\frac{1}{2}f+t_1-1} \frac{\Gamma(f+t_1+t_2-1-r)}{\Gamma(\frac{1}{2}f+t_2)\Gamma(\frac{1}{2}f+t_1-r)} q_{21}^{\frac{1}{2}f+t_2} p_{21}^{\frac{1}{2}f+t_1-1-r} \\ & \quad \left. I_{q_{31}^*}^{\frac{1}{2}f+t_3, f+t_1+t_2-1-r} \right\}. \quad (3.3.6) \end{aligned}$$

ii) t_2 is a non-negative integer

$$\begin{aligned}
& E(s_1^{2t_1} s_2^{2t_2} s_3^{2t_3} | A_0^n) P(A_0^n) \\
&= K_3 \sum_{r=0}^{\frac{1}{2}f+t_2-1} \frac{\Gamma(f+t_2+t_3-1-r)}{\Gamma(\frac{1}{2}f+t_3)\Gamma(\frac{1}{2}f+t_2-r)} q_{32}^{\frac{1}{2}f+t_3} p_{32}^{\frac{1}{2}f+t_2-1-r} \\
&\quad I_{p_{32}^*}(f+t_2+t_3-1-r, \frac{1}{2}f+t_1) \\
&- \sum_{r=0}^{\frac{1}{2}f+t_2-1} \frac{\Gamma(f+t_1+t_2-1-r)}{\Gamma(\frac{1}{2}f+t_1)\Gamma(\frac{1}{2}f+t_2-r)} p_{21}^{\frac{1}{2}f+t_1} q_{21}^{\frac{1}{2}f+t_2-1-r} \\
&\quad I_{q_{31}^*}(\frac{1}{2}f+t_3, f+t_1+t_2-1-r) \Big], \tag{3.3.7}
\end{aligned}$$

and

iii) t_3 is a non-negative integer

$$\begin{aligned}
& E(s_1^{2t_1} s_2^{2t_2} s_3^{2t_3} | A_0^n) P(A_0^n) \\
&= K_3 \left\{ I_{q_{21}}(\frac{1}{2}f+t_2, \frac{1}{2}f+t_1) \right. \\
&\quad - \sum_{r=0}^{\frac{1}{2}f+t_3-1} \frac{\Gamma(f+t_1+t_3-1-r)}{\Gamma(\frac{1}{2}f+t_1)\Gamma(\frac{1}{2}f+t_3-r)} p_{31}^{\frac{1}{2}f+t_1} q_{31}^{\frac{1}{2}f+t_3-1-r} \\
&\quad \left[I_{q_{21}}(\frac{1}{2}f+t_2, f+t_1+t_3-1-r) - I_{q_{21}^*}(\frac{1}{2}f+t_2, f+t_1+t_3-1-r) \right] \\
&\quad - \sum_{r=0}^{\frac{1}{2}f+t_3-1} \frac{\Gamma(f+t_2+t_3-1-r)}{\Gamma(\frac{1}{2}f+t_2)\Gamma(\frac{1}{2}f+t_3-r)} p_{32}^{\frac{1}{2}f+t_2} q_{32}^{\frac{1}{2}f+t_3-1-r} \\
&\quad \left. I_{p_{32}^*}(f+t_2+t_3-1-r, \frac{1}{2}f+t_1) \right\}, \tag{3.3.8}
\end{aligned}$$

where
$$K_3 = \prod_{i=1}^3 \Gamma(\frac{1}{2}f+t_i) / \Gamma(\frac{1}{2}f) (f/2\sigma_i^2)^{t_i} ,$$

$$\bar{q}_{21} = \lambda\theta_{32}/(\lambda+\theta_{31}+\lambda\theta_{32}) ,$$

$$q_{21}^* = \theta_{32}/(\lambda+\theta_{31}+\theta_{32}) ,$$

$$p_{32}^* = (\lambda+\theta_{32})/(\lambda+\theta_{31}+\theta_{32}) ,$$

and (3.3.9)

$$q_{31}^* = \lambda/(\lambda+\theta_{31}+\lambda\theta_{32}) .$$

Proof Since the characteristic y_i under study in the i th stratum is assumed to be normally distributed, $y_{i1}, y_{i2}, \dots, y_{im}$ can be considered as a random sample of m independent observations from a normal distribution with unknown mean μ_i and unknown variance σ_i^2 , $i=1,2,3$. It follows that fs_i^2/σ_i^2 is distributed as chi-square with $f=m-1$ degrees of freedom. Let $v_i=s_i^2$, $i=1,2,3$. Then the joint density of v_1, v_2 and v_3 is given by

$$f(v_1, v_2, v_3) = C_3 (v_1 v_2 v_3)^{\frac{1}{2}f-1} \exp(-\sum_{i=1}^3 f v_i / 2\sigma_i^2) \quad \text{if } v_i \geq 0, i=1,2,3, \quad (3.3.10)$$

$$= 0 \quad \text{otherwise,}$$

where
$$C_3 = \prod_{i=1}^3 [(f/2\sigma_i^2)^{\frac{1}{2}f} / \Gamma(\frac{1}{2}f)] . \quad (3.3.11)$$

Now,

$$\begin{aligned} E(s_1^{2t_1} s_2^{2t_2} s_3^{2t_3} | A_0'') P(A_0'') &= E(v_1^{t_1} v_2^{t_2} v_3^{t_3} | A_0'') P(A_0'') \\ &= \iiint_{A_0''} v_1^{t_1} v_2^{t_2} v_3^{t_3} f(v_1, v_2, v_3) dv_1 dv_2 dv_3 . \end{aligned} \quad (3.3.12)$$

We shall prove part iii) first. Part i) and ii) can be proved in a similar manner.

Assume that t_3 is a non-negative integer. The set A_0'' can be expressed as the union of two disjoint sets A_{01}'' and A_{02}'' , where A_{01}'' and A_{02}'' are defined by

$$A_{01}'' : \{0 < v_3 < \lambda v_1, \quad v_1 < v_2 < \lambda v_1, \quad 0 < v_1\},$$

(3.3.13)

and

$$A_{02}'' : \{0 < v_3 < \lambda v_2, \quad 0 < v_2 < v_1, \quad 0 < v_1\}.$$

Thus

$$E(v_1^{t_1} v_2^{t_2} v_3^{t_3} | A_0'') P(A_0'') = I_{01}'' + I_{02}'' , \quad (3.3.14)$$

where

$$I_{0i}'' = \iiint_{A_{0i}''} v_1^{t_1} v_2^{t_2} v_3^{t_3} f(v_1, v_2, v_3) dv_1 dv_2 dv_3, \quad i=1,2. \quad (3.3.15)$$

Now,

$$I_{01}'' = c_3 \int_0^\infty \int_{v_1}^{\lambda v_1} \int_0^{\lambda v_1} v_1^{\frac{1}{2}f+t_1-1} v_2^{\frac{1}{2}f+t_2-1} v_3^{\frac{1}{2}f+t_3-1} \exp(-\frac{1}{2}f \sum_{i=1}^3 v_i / \sigma_i^2) dv_3 dv_2 dv_1 . \quad (3.3.16)$$

Using the fact that when h is a non-negative integer,

$$-\int_u^\infty x^h e^{-ax} dx = \frac{\Gamma(h+1)}{a^{h+1}} \sum_{r=0}^h e^{-au} (au)^{h-r} / \Gamma(h+1-r) , \quad (3.3.17)$$

the integral with respect to v_3 can be expressed as a finite sum. Then integration and summation can be interchanged.

Hence, we obtain

$$I''_{01} = c_3 [\Gamma(\frac{1}{2}f+t_3)/(f/2\sigma_3^2)^{\frac{1}{2}f+t_3}] \\ [I - \sum_{r=0}^{\frac{1}{2}f+t_3-1} (f\lambda/2\sigma_3^2)^{\frac{1}{2}f+t_3-1-r} I_r / \Gamma(\frac{1}{2}f+t_3-r)] , \quad (3.3.18)$$

where

$$I = \int_0^\infty \int_{v_1}^{\lambda v_1} v_1^{\frac{1}{2}f+t_1-1} v_2^{\frac{1}{2}f+t_2-1} \exp\left\{-\frac{1}{2}f\left[(v_1/\sigma_1^2)+(v_2/\sigma_2^2)\right]\right\} dv_2 dv_1, \quad (3.3.19)$$

and

$$I_r = \int_0^\infty \int_{v_1}^{\lambda v_1} v_1^{f+t_1+t_3-2-r} v_2^{\frac{1}{2}f+t_2-1} \\ \exp\left\{-\frac{1}{2}f\left[\left(\frac{1}{\sigma_1^2} + \frac{\lambda}{\sigma_3^2}\right)v_1 + (v_2/\sigma_2^2)\right]\right\} dv_2 dv_1. \quad (3.3.20)$$

From 3.3.19,

$$I = \int_0^\infty \int_0^{\lambda v_1} v_1^{\frac{1}{2}f+t_1-1} v_2^{\frac{1}{2}f+t_2-1} \exp\left\{-\frac{1}{2}f\left[(v_1/\sigma_1^2)+(v_2/\sigma_2^2)\right]\right\} dv_2 dv_1 \\ \int_0^\infty \int_0^{v_1} v_1^{\frac{1}{2}f+t_1-1} v_2^{\frac{1}{2}f+t_2-1} \exp\left\{-\frac{1}{2}f\left[(v_1/\sigma_1^2)+(v_2/\sigma_2^2)\right]\right\} dv_2 dv_1 \\ = [\Gamma(\frac{1}{2}f)\Gamma(\frac{1}{2}f)/(f/2\sigma_1^2)^{\frac{1}{2}f}(f/2\sigma_2^2)^{\frac{1}{2}f}] \\ [E(v_1^{t_1} v_2^{t_2} | v_2/v_1 < \lambda) P(v_2/v_1 < \lambda) - E(v_1^{t_1} v_2^{t_2} | v_2/v_1 < 1) P(v_2/v_1 < 1)]. \quad (3.3.21)$$

Letting $f_1=f_2=f$ in 3.1.7, we obtain the first expectation and, further, letting $\lambda=1$, we obtain the second expectation in 3.3.21. Thus,

$$\begin{aligned}
I &= \left[\prod_{i=1}^2 \left(\frac{\Gamma(\frac{1}{2}f) \Gamma(\frac{1}{2}f)}{(f/2\sigma_1^2)^{\frac{1}{2}f} (f/2\sigma_2^2)^{\frac{1}{2}f}} \right) \right] \prod_{i=1}^2 \left[(2\sigma_i^2/f)^{t_i} \Gamma(\frac{1}{2}f+t_i) / \Gamma(\frac{1}{2}f) \right] \\
&\quad \left[I_{q_{21}}(\frac{1}{2}f+t_2, \frac{1}{2}f+t_1) - I_{1/(\theta_{21}+1)}(\frac{1}{2}f+t_2, \frac{1}{2}f+t_1) \right] \\
&= \prod_{i=1}^2 \left[\Gamma(\frac{1}{2}f+t_i) / (f/2\sigma_i^2)^{\frac{1}{2}f+t_i} \right] \left[I_{q_{21}}(\frac{1}{2}f+t_2, \frac{1}{2}f+t_1) \right. \\
&\quad \left. - I_{1/(\theta_{21}+1)}(\frac{1}{2}f+t_2, \frac{1}{2}f+t_1) \right]. \tag{3.3.22}
\end{aligned}$$

If we make the transformation

$$[(1/\sigma_1^2) + (\lambda/\sigma_3^2)] v_1 = v_1' / \sigma_1^2, \tag{3.3.23}$$

and follow the same procedure as before, we obtain

$$\begin{aligned}
I_r &= p_{31}^{f+t_1+t_3-1-r} \left\{ \int_0^\infty \int_0^\infty \lambda p_{31}^{v_1'} \frac{v_1'^{f+t_1+t_3-2-r}}{v_1'} \frac{v_2^{\frac{1}{2}f+t_2-1}}{v_2} \right. \\
&\quad \exp \left[-\frac{1}{2}f \left(\frac{v_1'}{\sigma_1^2} + \frac{v_2}{\sigma_2^2} \right) \right] dv_2 dv_1' - \int_0^\infty \int_0^\infty p_{31}^{v_1'} \frac{v_1'^{f+t_1+t_3-2-r}}{v_1'} \frac{v_2^{\frac{1}{2}f+t_2-1}}{v_2} \\
&\quad \exp \left[-\frac{1}{2}f \left(\frac{v_1'}{\sigma_1^2} + \frac{v_2}{\sigma_2^2} \right) \right] dv_2 dv_1' \Big\} \\
&= p_{31}^{f+t_1+t_3-1-r} \left[\prod_{i=1}^2 \left(\frac{\Gamma(\frac{1}{2}f) \Gamma(\frac{1}{2}f)}{(f/2\sigma_1^2)^{\frac{1}{2}f} (f/2\sigma_2^2)^{\frac{1}{2}f}} \right) \right] \\
&\quad \left[E \left(v_1'^{\frac{1}{2}f+t_1+t_3-1-r} v_2^{t_2} \mid v_2/v_1' < \lambda p_{31} \right) P(v_2/v_1' < \lambda p_{31}) \right. \\
&\quad \left. - E \left(v_1'^{\frac{1}{2}f+t_1+t_3-1-r} v_2^{t_2} \mid v_2/v_1' < p_{31} \right) P(v_2/v_1' < p_{31}) \right]. \tag{3.3.24}
\end{aligned}$$

Letting $t_1 = \frac{1}{2}f+t_1+t_3-1-r$ and $\lambda = \lambda p_{31}$ in 3.1.7, we obtain

the first expectation and letting $\lambda = p_{31}$, we obtain the second expectation in 3.3.24. Thus,

$$\begin{aligned}
I_r = p_{31}^{f+t_1+t_3-1-r} & \left[\prod (f+t_1+t_3-1-r) \prod (\tfrac{1}{2}f+t_2) / \right. \\
& \left. (f/2\sigma_1^2)^{f+t_1+t_3-1-r} (f/2\sigma_2^2)^{\tfrac{1}{2}f+t_2} \right] \\
& \left[I_{q_{21}}^{-}(\tfrac{1}{2}f+t_2, f+t_1+t_3-1-r) - I_{q_{21}}^*(\tfrac{1}{2}f+t_2, f+t_1+t_3-1-r) \right].
\end{aligned} \tag{3.3.25}$$

Substituting from 3.3.22 and 3.3.25 in 3.3.18, we obtain, after simplification,

$$\begin{aligned}
I_{01}'' = K_3 \{ & I_{q_{21}}(\tfrac{1}{2}f+t_2, \tfrac{1}{2}f+t_1) - I_1/(\theta_{21}+1)(\tfrac{1}{2}f+t_2, \tfrac{1}{2}f+t_1) \\
& - \sum_{r=0}^{\tfrac{1}{2}f+t_3-1} \left[\prod (f+t_1+t_3-1-r) / \prod (\tfrac{1}{2}f+t_1) \prod (\tfrac{1}{2}f+t_3-r) \right] p_{31}^{\tfrac{1}{2}f+t_1} q_{31}^{\tfrac{1}{2}f+t_3-1-r} \\
& \left. \left[I_{q_{21}}^{-}(\tfrac{1}{2}f+t_2, f+t_1+t_3-1-r) - I_{q_{21}}^*(\tfrac{1}{2}f+t_2, f+t_1+t_3-1-r) \right] \right\}.
\end{aligned} \tag{3.3.26}$$

Next, consider

$$\begin{aligned}
I_{02}'' = C_3 \int_0^\infty \int_0^{v_1} \int_0^{v_2} & v_1^{\lambda v_2} v_2^{\tfrac{1}{2}f+t_1-1} v_3^{\tfrac{1}{2}f+t_2-1} v_3^{\tfrac{1}{2}f+t_3-1} \\
& \exp(-\tfrac{1}{2}f \sum_{i=1}^3 v_i / \sigma_i^2) dv_3 dv_2 dv_1.
\end{aligned} \tag{3.3.27}$$

Following the same technique as before, we obtain

$$\begin{aligned}
I_{02}'' = K_3 \{ & I_1/(\theta_{21}+1)(\tfrac{1}{2}f+t_2, \tfrac{1}{2}f+t_1) \\
& - \sum_{r=0}^{\tfrac{1}{2}f+t_3-1} \left[\prod (f+t_2+t_3-1-r) / \prod (\tfrac{1}{2}f+t_2) \prod (\tfrac{1}{2}f+t_3-r) \right] \\
& p_{32}^{\tfrac{1}{2}f+t_2} q_{32}^{\tfrac{1}{2}f+t_3-1-r} I_{p_{32}^*}^*(f+t_2+t_3-1-r, \tfrac{1}{2}f+t_1) \right\}.
\end{aligned} \tag{3.3.28}$$

Then we obtain 3.3.8 by adding 3.3.26 and 3.3.28.

To prove i), we assume that t_1 is a non-negative integer. The set A_0'' can be expressed as the union of two disjoint sets A_{03}'' and A_{04}'' defined as follows:

$$A_{03}'' : \{v_3/\lambda < v_1 < \infty, \quad v_2 < v_3 < \lambda v_2, \quad 0 < v_2\},$$

(3.3.29)

and

$$A_{04}'' : \{v_2/\lambda < v_1 < \infty, \quad 0 < v_3 < v_2, \quad 0 < v_2\}.$$

Thus

$$E(s_1^{2t_1} s_2^{2t_2} s_3^{2t_3} | A_0'') P(A_0'') = I_{03}'' + I_{04}'' ,$$

where

$$\begin{aligned} I_{03}'' &= \iiint_{A_{03}''} v_1^{t_1} v_2^{t_2} v_3^{t_3} f(v_1, v_2, v_3) dv_1 dv_2 dv_3 \\ &= C_3 \int_0^\infty \int_{v_2}^{\lambda v_2} \int_{v_3/\lambda}^\infty v_1^{\frac{1}{2}f+t_1-1} v_2^{\frac{1}{2}f+t_2-1} v_3^{\frac{1}{2}f+t_3-1} \\ &\quad \exp(-\frac{1}{2}f \sum_{i=1}^3 v_i / \sigma_i^2) dv_1 dv_3 dv_2 , \end{aligned} \quad (3.3.30)$$

and

$$\begin{aligned} I_{04}'' &= \iiint_{A_{04}''} v_1^{t_1} v_2^{t_2} v_3^{t_3} f(v_1, v_2, v_3) dv_1 dv_2 dv_3 \\ &= C_3 \int_0^\infty \int_0^{v_2} \int_{v_2/\lambda}^\infty v_1^{\frac{1}{2}f+t_1-1} v_2^{\frac{1}{2}f+t_2-1} v_3^{\frac{1}{2}f+t_3-1} \\ &\quad \exp(-\frac{1}{2}f \sum_{i=1}^3 v_i / \sigma_i^2) dv_1 dv_3 dv_2 . \end{aligned} \quad (3.3.31)$$

Since $\frac{1}{2}f+t_1-1$ is a non-negative integer, the integrals with respect to v_1 , in 3.3.30 and 3.3.31, can be expressed as finite sums by using 3.3.17. Following the same technique as

before, we obtain

$$\begin{aligned}
 I_{03}'' &= K_3 \sum_{r=0}^{\frac{1}{2}f+t_1-1} \left[\prod (f+t_1+t_3-1-r) / \prod (\frac{1}{2}f+t_3) \prod (\frac{1}{2}f+t_1-r) \right] \\
 &\quad q_{31}^{\frac{1}{2}f+t_3} p_{31}^{\frac{1}{2}f+t_1-1-r} \left[I_{q_{21}} (\frac{1}{2}f+t_2, f+t_1+t_3-1-r) \right. \\
 &\quad \left. - I_{q_{21}^*} (\frac{1}{2}f+t_2, f+t_1+t_3-1-r) \right], \quad (3.3.32)
 \end{aligned}$$

and

$$\begin{aligned}
 I_{04}'' &= K_3 \sum_{r=0}^{\frac{1}{2}f+t_1-1} \left[\prod (f+t_1+t_2-1-r) / \prod (\frac{1}{2}f+t_2) \prod (\frac{1}{2}f+t_1-r) \right] \\
 &\quad q_{21}^{\frac{1}{2}f+t_2} p_{21}^{\frac{1}{2}f+t_1-1-r} I_{q_{31}^*} (\frac{1}{2}f+t_3, f+t_1+t_2-1-r). \quad (3.3.33)
 \end{aligned}$$

Then we obtain 3.3.6 by adding 3.3.32 and 3.3.33. Finally, we assume that t_2 is a non-negative integer. The set A_0'' can be expressed as

$$A_0'' : \{v_3/\lambda < v_2 < \lambda v_1, \quad 0 < v_3 < \lambda v_1, \quad 0 < v_1\}. \quad (3.3.34)$$

Then

$$\begin{aligned}
 &E(s_1^{2t_1} s_2^{2t_2} s_3^{2t_3} | A_0'') P(A_0'') \\
 &= \iiint_{A_0''} v_1^{t_1} v_2^{t_2} v_3^{t_3} f(v_1, v_2, v_3) dv_1 dv_2 dv_3 \\
 &= C_3 \int_0^\infty \int_0^{\lambda v_1} \int_{v_3/\lambda}^{\lambda v_1} v_1^{\frac{1}{2}f+t_1-1} v_2^{\frac{1}{2}f+t_2-1} v_3^{\frac{1}{2}f+t_3-1} \\
 &\quad \exp(-\frac{1}{2}f \sum_{i=1}^3 v_i / \sigma_i^2) dv_2 dv_3 dv_1. \quad (3.3.35)
 \end{aligned}$$

Since $\frac{1}{2}f+t_2-1$ is a non-negative integer, we can express the integral with respect to v_2 , using 3.3.17, as a finite sum.

Following the same technique as before, we obtain 3.3.7 from 3.3.35. Q.E.D.

Using Lemma 3.1 or otherwise, it can be seen that

$$E(s_j/s_i) = \frac{1}{2}G\theta_{ji}^{\frac{1}{2}}. \quad (3.3.36)$$

Also,

$$E(s_j/s_i | A_1^n)P(A_1^n) = \frac{1}{2}G\theta_{ji}^{\frac{1}{2}} - E(s_j/s_i | A_0^n)P(A_0^n). \quad (3.3.37)$$

Thus, letting $t_1 = -\frac{1}{2}$, $t_2 = \frac{1}{2}$ and $t_3 = 0$ in 3.3.8 and substituting in 3.3.37, we get

$$\begin{aligned} & E(s_2/s_1 | A_1^n)P(A_1^n) \\ &= \frac{1}{2}G\theta_{21}^{\frac{1}{2}} - \frac{1}{2}G\theta_{21}^{\frac{1}{2}} \left\{ I_{q_{21}} \left(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2} \right) \right. \\ &\quad - \sum_{r=0}^{\frac{1}{2}f-1} \left[\Gamma(f - \frac{3}{2} - r) / \Gamma(\frac{1}{2}f - \frac{1}{2}) \Gamma(\frac{1}{2}f - r) \right] p_{31}^{\frac{1}{2}f-1} q_{31}^{\frac{1}{2}f-1-r} \\ &\quad \left[I_{q_{21}} \left(\frac{1}{2}f + \frac{1}{2}, f - \frac{3}{2} - r \right) - I_{q_{21}^*} \left(\frac{1}{2}f + \frac{1}{2}, f - \frac{3}{2} - r \right) \right] \\ &\quad \left. - \sum_{r=0}^{\frac{1}{2}f-1} \left[\Gamma(f - \frac{1}{2} - r) / \Gamma(\frac{1}{2}f + \frac{1}{2}) \Gamma(\frac{1}{2}f - r) \right] p_{32}^{\frac{1}{2}f+1} q_{32}^{\frac{1}{2}f-1-r} I_{p_{32}^*} \left(f - \frac{1}{2} - r, \frac{1}{2}f - \frac{1}{2} \right) \right\}. \end{aligned}$$

Similarly, using Lemma 3.3 and proper choice of t_1 , t_2 and t_3 , we can get $E(s_j/s_i | A_1^n)P(A_1^n)$ for all $i \neq j$. By letting $t_1 = t_2 = t_3 = 0$ in 3.3.7, we obtain

$$\begin{aligned} P(A_0^n) &= \sum_{r=0}^{\frac{1}{2}f-1} \binom{f-2-r}{\frac{1}{2}f-1-r} \left[q_{32}^{\frac{1}{2}f} p_{32}^{\frac{1}{2}f-1-r} I_{p_{32}^*} (f-1-r, \frac{1}{2}f) \right. \\ &\quad \left. - p_{21}^{\frac{1}{2}f} q_{21}^{\frac{1}{2}f-1-r} I_{q_{31}^*} \left(\frac{1}{2}f, f-1-r \right) \right]. \end{aligned} \quad (3.3.38)$$

$$\text{Hence } P(A_1^n) = 1 - P(A_0^n). \quad (3.3.39)$$

Using the above results in 3.3.5 and simplifying, we obtain

$$\begin{aligned}
& E_1[V(\bar{y}_w | A_1'')] \\
&= \sigma_2^2(w_1^2/\theta_{21} + w_2^2 + w_3^2\theta_{32})/n - \sigma_2^2(w_1/\theta_{21} + w_2 + w_3\theta_{32})/N \\
&+ \frac{\sigma_2^2}{n} G[w_1 w_2 I_{p_{21}}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f - \frac{1}{2})/\theta_{21}^{\frac{1}{2}} + w_1 w_3 \theta_{32}^{\frac{1}{2}}/\theta_{21}^{\frac{1}{2}} + w_2 w_3 \theta_{32}^{\frac{1}{2}} \\
&+ \frac{\frac{1}{2}f-1}{r=0} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} A(\lambda, \theta_{21}, \theta_{32}) \\
&+ \frac{\frac{1}{2}f-1}{r=0} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} B(\lambda, \theta_{21}, \theta_{32})] / P(A_1'') , \\
\end{aligned} \tag{3.3.40}$$

where

$$\begin{aligned}
& A(\lambda, \theta_{21}, \theta_{32}) \\
&= (w_1 w_2 p_{31}^{\frac{1}{2}f-\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f-\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r}) \\
&[I_{q_{21}}(\frac{1}{2}f+\frac{1}{2}, f-3/2-r) - I_{q_{21}^*}(\frac{1}{2}f+\frac{1}{2}, f-3/2-r)] \\
&+ (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f-\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f-\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r}) \\
&I_{q_{31}^*}(\frac{1}{2}f+\frac{1}{2}, f-3/2-r) \\
&+ (w_1 w_2 p_{32}^{\frac{1}{2}f-\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f-\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}}) \\
&I_{p_{32}^*}(f-3/2-r, \frac{1}{2}f+\frac{1}{2}) , \\
\end{aligned} \tag{3.3.41}$$

and

$$\begin{aligned}
& B(\lambda, \theta_{21}, \theta_{32}) \\
&= (w_1 w_2 p_{31}^{\frac{1}{2}f+\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f+\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r}) \\
&\quad [I_{q_{21}}^{\frac{1}{2}f-\frac{1}{2}, f-\frac{1}{2}-r} - I_{q_{21}^*}^{\frac{1}{2}f-\frac{1}{2}, f-\frac{1}{2}-r}] \\
&\quad + (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f+\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f+\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r}) \\
&\quad I_{q_{31}^*}^{\frac{1}{2}f-\frac{1}{2}, f-\frac{1}{2}-r} \\
&\quad + (w_1 w_2 p_{32}^{\frac{1}{2}f+\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f+\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}}) \\
&\quad I_{p_{32}^*}^{\frac{1}{2}f-\frac{1}{2}-r, \frac{1}{2}f-\frac{1}{2}} . \tag{3.3.42}
\end{aligned}$$

Using the above results and 3.3.4 in 3.3.3, we obtain

$$\begin{aligned}
V(\bar{y}_w)_S &= \sigma_2^2(w_1^2/\theta_{21} + w_2^2 + w_3^2\theta_{32})/n - \sigma_2^2(w_1/\theta_{21} + w_2 + w_3\theta_{32})/N \\
&\quad + \sigma_2^2[w_1(1-w_1)/\theta_{21} + w_2(1-w_2) + w_3(1-w_3)\theta_{32}]P(A_0'')/n \\
&\quad + \sigma_2^2 G[w_1 w_2 I_{p_{21}}^{\frac{1}{2}f-\frac{1}{2}, \frac{1}{2}f-\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} + w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} + w_2 w_3 \theta_{32}^{\frac{1}{2}}] / n \\
&\quad + \frac{\sigma_2^2}{n} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{3}{2}-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} A(\lambda, \theta_{21}, \theta_{32}) \\
&\quad + \frac{\sigma_2^2}{n} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} B(\lambda, \theta_{21}, \theta_{32}) . \tag{3.3.43}
\end{aligned}$$

It can be verified easily that

$$\lim_{\lambda \rightarrow \infty} P(A_0'') = 1 ,$$

$$\begin{aligned}
&\lim_{\lambda \rightarrow \infty} [w_1 w_2 I_{p_{21}}^{\frac{1}{2}f-\frac{1}{2}, \frac{1}{2}f-\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} + w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} + w_2 w_3 \theta_{32}^{\frac{1}{2}}] \\
&= w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} + w_2 w_3 \theta_{32}^{\frac{1}{2}} ,
\end{aligned}$$

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} \sum_{r=0}^{\frac{1}{2}f-1} \frac{\prod(f - \frac{3}{2} - r)}{\prod(\frac{1}{2}f - \frac{1}{2}) \prod(\frac{1}{2}f - r)} A(\lambda, \theta_{21}, \theta_{32}) \\
= -w_2 w_3 \theta_{32}^{\frac{1}{2}} [I_{\theta_{32}/\theta_{32}+1}(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2}) + I_{1/\theta_{32}+1}(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2})] \\
-w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} \sum_{r=0}^{\frac{1}{2}f-1} \frac{\prod(f - \frac{1}{2} - r)}{\prod(\frac{1}{2}f + \frac{1}{2}) \prod(\frac{1}{2}f - r)} B(\lambda, \theta_{21}, \theta_{32}) \\
= -w_2 w_3 \theta_{32}^{\frac{1}{2}} [I_{\theta_{32}/\theta_{32}+1}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2}) + I_{1/\theta_{32}+1}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2})] \\
-w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}}.
\end{aligned}$$

If now we let λ tend to infinity in 3.3.43, it can be seen that $V(\bar{y}_w)_S$ reduces to $V(\bar{y}_w)_P$ as is to be expected, i.e.,

$$V(\bar{y}_w)_P = (\frac{1}{n} - \frac{1}{N}) \sigma_2^2 (w_1/\theta_{21} + w_2 + w_3 \theta_{32}). \quad (3.3.44)$$

Putting $\lambda=0$, in 3.3.43, we obtain the variance of \bar{y}_w under modified Neyman allocation, i.e.,

$$\begin{aligned}
V(\bar{y}_w)_N = \sigma_2^2 (w_1^2/\theta_{21} + w_2^2 + w_3^2 \theta_{32})/n - \sigma_2^2 (w_1/\theta_{21} + w_2 + w_3 \theta_{32})/N \\
+ \sigma_2^2 G(w_1 w_2 / \theta_{21}^{\frac{1}{2}} + w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} + w_2 w_3 \theta_{32}^{\frac{1}{2}}) / n. \quad (3.3.45)
\end{aligned}$$

From the above discussion, we can see that the evaluation of $V(\bar{y}_w)_S$ consists in the computation of expectations of the type

$$E\left(\prod_{i=1}^k s_i^{2t_i} \mid A_0\right)$$

where A_0 represents the set in which proportional allocation

is adopted. We have already seen that, even for $k=3$, the expectations are very complicated. Thus for $k>3$ the expectations are likely to be still more complicated. If we assume again that f is an even integer and apply 3.3.17 repeatedly, the expectation $E(\prod_{i=1}^k s_i^{2t_i} | A_0)$ could be reduced to a stage where Lemma 3.3 is applicable. Hence it is possible to evaluate $V(\bar{y}_w)_S$ for any k in general. However, the results will be extremely complicated.

IV. COMPARISON OF DIFFERENT ALLOCATIONS

In this section, we shall compare sometimes proportional allocation with proportional allocation and with modified Neyman allocation. We shall first consider the case $k=2$ and then extend the results further.

A. Comparison of Sometimes Proportional Allocation with Proportional Allocation for $k=2$ and $\sigma_1^2 \leq \sigma_2^2$

Consider the difference function

$$\begin{aligned} D_1(\lambda, \theta_{21}) &= [V(\bar{y}_w)_P - V(\bar{y}_w)_S] / \frac{w_1 w_2}{n} \sigma_1^2 \\ &= (1 + \theta_{21}) I_{p_{21}}(\tfrac{1}{2}f, \tfrac{1}{2}f) - G \theta_{21}^{\frac{1}{2}} I_{p_{21}}(\tfrac{1}{2}f - \tfrac{1}{2}, \tfrac{1}{2}f - \tfrac{1}{2}), \quad (4.1.1) \end{aligned}$$

where

$$p_{21} = 1 - q_{21} = \theta_{21} / (\theta_{21} + \lambda). \quad (4.1.2)$$

Clearly, if $D_1(\lambda, \theta_{21}) > 0$ sometimes proportional allocation is more efficient than proportional allocation.

If we let

$$I_i(p_{21}) = I_{p_{21}}(\tfrac{1}{2}f + i, \tfrac{1}{2}f + i), \quad (4.1.3)$$

then $D_1(\lambda, \theta_{21})$ can be written as

$$D_1(\lambda, \theta_{21}) = (1 + \theta_{21}) I_0(p_{21}) - G \theta_{21}^{\frac{1}{2}} I_{-\frac{1}{2}}(p_{21}). \quad (4.1.4)$$

Let $Q_i(p_{21})$ and $R_i(p_{21})$ denote the partial derivative of $I_i(p_{21})$ with respect to λ and θ_{21} respectively. Then

$$\begin{aligned}
Q_i(p_{21}) &= \frac{\partial}{\partial \lambda} I_i(p_{21}) \\
&= \frac{\partial}{\partial \lambda} \int_0^{p_{21}} x^{\frac{1}{2}f+i-1} (1-x)^{\frac{1}{2}f+i-1} dx / B(\frac{1}{2}f+i, \frac{1}{2}f+i) \\
&= -p_{21}^{\frac{1}{2}f+i} q_{21}^{\frac{1}{2}f+i} / \lambda B(\frac{1}{2}f+i, \frac{1}{2}f+i). \quad (4.1.5)
\end{aligned}$$

And similarly,

$$R_i(p_{21}) = \frac{\partial}{\partial \theta_{21}} I_i(p_{21}) = p_{21}^{\frac{1}{2}f+i} q_{21}^{\frac{1}{2}f+i} / \theta_{21} B(\frac{1}{2}f+i, \frac{1}{2}f+i). \quad (4.1.6)$$

As θ_{21} tends to 1, we have the following lemma concerning the behavior of $D_1(\lambda, \theta_{21})$.

Lemma 4.1 For any $\lambda \geq 0$, $\lim_{\theta_{21} \rightarrow 1} D_1(\lambda, \theta_{21}) \leq 0$.

Proof From 4.1.4

$$\lim_{\theta_{21} \rightarrow 1} D_1(\lambda, \theta_{21}) = D_1(\lambda, 1) = 2I_0\left(\frac{1}{1+\lambda}\right) - GI_{-\frac{1}{2}}\left(\frac{1}{1+\lambda}\right).$$

Since for any $\theta_{21} \geq 0$,

$$\lim_{\lambda \rightarrow 0} I_i(p_{21}) = 1,$$

it can be seen that

$$\lim_{\lambda \rightarrow 0} D_1(\lambda, 1) = 2-G.$$

It can be shown that $G > 2$.

$$\therefore \lim_{\lambda \rightarrow 0} D_1(\lambda, 1) < 0.$$

Similarly, since for any $\theta_{21} \geq 0$,

$$\lim_{\lambda \rightarrow \infty} I_i(p_{21}) = 0^+,$$

it can be seen that

$$\lim_{\lambda \rightarrow \infty} D_1(\lambda, 1) = (2-G)0^+ = 0^-.$$

Again, using 4.1.5

$$\begin{aligned}\frac{\partial}{\partial \lambda} D_1(\lambda, 1) &= 2Q_0\left(\frac{1}{1+\lambda}\right) - GQ_{-\frac{1}{2}}\left(\frac{1}{1+\lambda}\right) \\ &= \lambda^{\frac{1}{2}f-3/2} (1-\lambda^{\frac{1}{2}})^2 / (1+\lambda)^f B(\frac{1}{2}f, \frac{1}{2}f) \geq 0.\end{aligned}$$

Then $D_1(\lambda, 1)$ tends to a negative quantity as λ tends to zero and tends to zero from below as λ tends to ∞ . Furthermore, $D_1(\lambda, 1)$ is a non-decreasing function in λ . Therefore

$$\lim_{\theta_{21} \rightarrow 1} D_1(\lambda, \theta_{21}) \leq 0 \quad \text{for any } \lambda \geq 0.$$

This completes the proof.

Next, let us consider the case that λ is a fixed but arbitrary non-negative number. We shall discuss the behavior of $D_1(\lambda, \theta_{21})$ as θ_{21} varies by first proving the following lemma.

Lemma 4.2 For any given $\lambda \geq 0$ satisfying

$$1 - \frac{1}{2}G + \lambda^{\frac{1}{2}f-1/2} (\lambda^{\frac{1}{2}} - 1) / B(\frac{1}{2}f, \frac{1}{2}f) > 0, \quad (4.1.7)$$

$\exists \theta'$ such that

$$\frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}) > 0 \quad \text{for each } \theta_{21} > \theta'.$$

Proof From 4.1.4, we have

$$\begin{aligned}\frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}) &= I_0(p_{21}) - \frac{1}{2}G\theta_{21}^{-\frac{1}{2}} I_{-\frac{1}{2}}(p_{21}) \\ &\quad + (1+\theta_{21})R_0(p_{21}) - G\theta_{21}^{\frac{1}{2}} R_{-\frac{1}{2}}(p_{21}) \\ &= I_0(p_{21}) - \frac{1}{2}G\theta_{21}^{-\frac{1}{2}} I_{-\frac{1}{2}}(p_{21}) \\ &\quad + \theta_{21}^{\frac{1}{2}f-1} \lambda^{\frac{1}{2}f-1/2} (\lambda^{\frac{1}{2}} - 1) (\theta_{21} - \lambda^{\frac{1}{2}}) / (\theta_{21} + \lambda)^f B(\frac{1}{2}f, \frac{1}{2}f).\end{aligned} \quad (4.1.8)$$

Noting that

$$\lim_{\theta_{21} \rightarrow \infty} I_i(p_{21})/\theta_{21}^h = 0^+ \quad \text{for } h > 0, \quad (4.1.9)$$

we find that

$$\lim_{\theta_{21} \rightarrow \infty} \frac{1}{\theta_{21}} \frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}) = [1 - \frac{1}{2}G + \lambda^{\frac{1}{2}f - \frac{1}{2}}(\lambda^{\frac{1}{2}} - 1)/B(\frac{1}{2}f, \frac{1}{2}f)] 0^+ = 0^+.$$

This implies that $\exists \theta'$ such that

$$\frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}) > 0 \quad \text{for each } \theta_{21} > \theta'. \quad \text{Q.E.D.}$$

After the above two lemmas are established, we are ready to prove the following theorem.

Theorem 4.1 For any $\lambda \geq 0$ satisfying 4.1.7,

$\exists \theta_0$ such that

$$D_1(\lambda, \theta_0) = 0$$

and

$$D_1(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \theta_{21} \geq \theta_0.$$

Proof From Lemma 4.1, we know

$$\lim_{\theta_{21} \rightarrow 1} D_1(\lambda, \theta_{21}) \geq 0 \quad \text{for any } \lambda \geq 0.$$

And from Lemma 4.2, $\exists \theta'$ such that

$$\frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}) > 0 \quad \text{for each } \theta_{21} > \theta',$$

i.e., for each $\theta_{21} > \theta'$, $D_1(\lambda, \theta_{21})$ is an increasing function in θ_{21} . Also it can be verified that

$$\lim_{\theta_{21} \rightarrow \infty} \frac{D_1(\lambda, \theta_{21})}{\theta_{21}} = 1.$$

Therefore, $\exists \theta_0$ such that

$$D_1(\lambda, \theta_0) = 0$$

and

$$D_1(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \theta_{21} \geq \theta_0.$$

This completes the proof of Theorem 4.1.

For any given $\lambda \geq 0$ satisfying 4.1.7, Theorem 4.1 assures us that there exists a θ_0 such that for each $\theta_{21} \geq \theta_0$ sometimes proportional allocation is always more efficient than proportional allocation.

Taking, in particular, $\lambda=0$, we obtain the following corollary to Theorem 4.1.

$$\text{Corollary 4.1} \quad D_1(0, \theta_{21}) > 0 \quad \text{if } \theta_{21} > \theta_m$$

and

$$D_1(0, \theta_{21}) \leq 0 \quad \text{if } 1 \leq \theta_{21} \leq \theta_m,$$

$$\text{where } \theta_m = \frac{1}{2}(G^2 - 2 + G\sqrt{G^2 - 4}) \quad (4.1.10)$$

Proof When $\lambda=0$, from 4.1.2, $p_{21}=1$. Then both $I_0(p_{21})$ and $I_{-\frac{1}{2}}(p_{21})$, in 4.1.4 are equal to 1. Therefore

$$D_1(0, \theta_{21}) = \theta_{21} - G\theta_{21}^{\frac{1}{2}} + 1,$$

which is a quadratic function in $\theta_{21}^{\frac{1}{2}}$. Since $\theta_{21} \geq 1$,

$D_1(0, \theta_{21})=0$ when $\theta_{21} = \frac{1}{2}(G^2 - 2 + G\sqrt{G^2 - 4}) = \theta_m$. Thus from Theorem 4.1

$$D_1(0, \theta_{21}) > 0 \quad \text{if } \theta_{21} > \theta_m$$

and

$$D_1(0, \theta_{21}) \leq 0 \quad \text{if } 1 \leq \theta_{21} \leq \theta_m.$$

Corollary 4.1 tells that when $\lambda=0$ sometimes proportional

allocation is more efficient than proportional allocation if θ_{21} is larger than θ_m and proportional allocation is more efficient otherwise. We should note that when $\lambda=0$, the event A_0 , defined in 3.1.2, is a null event. Hence sometimes proportional allocation reduces to modified Neyman allocation. This is, of course, a limiting case of sometimes proportional allocation. Hence Corollary 4.1 can also be considered as a corollary concerning the comparison between proportional allocation and modified Neyman allocation. Modified Neyman allocation is more efficient than proportional allocation if $\theta_{21} > \theta_m$ and proportional allocation is more efficient otherwise.

In Figure 4.1, we present several graphs to demonstrate the behavior of $D_1(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=0.5$.

We shall now consider the case that θ_{21} is a fixed but arbitrary number. Since we assume $\sigma_1^2 \leq \sigma_2^2$, $\theta_{21} \geq 1$. We shall now prove a theorem analogous to Theorem 4.1 so that for a given θ_{21} there exists a λ_0 such that for each $\lambda < \lambda_0$ sometimes proportional allocation is always more efficient than proportional allocation.

From 4.1.5, we have

$$\begin{aligned} \frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}) &= (1 + \theta_{21}) Q_0(p_{21}) - G \theta_{21}^{\frac{1}{2}} Q_{-\frac{1}{2}}(p_{21}) \\ &= \theta_{21}^{\frac{1}{2}} \lambda^{\frac{1}{2}} \Gamma^{-3/2} (\lambda^{\frac{1}{2}} - 1) (\lambda^{\frac{1}{2}} - \theta_{21}) / (\theta_{21} + \lambda)^{\frac{1}{2}} B(\frac{1}{2}\Gamma, \frac{1}{2}\Gamma). \end{aligned}$$

(4.1.11)

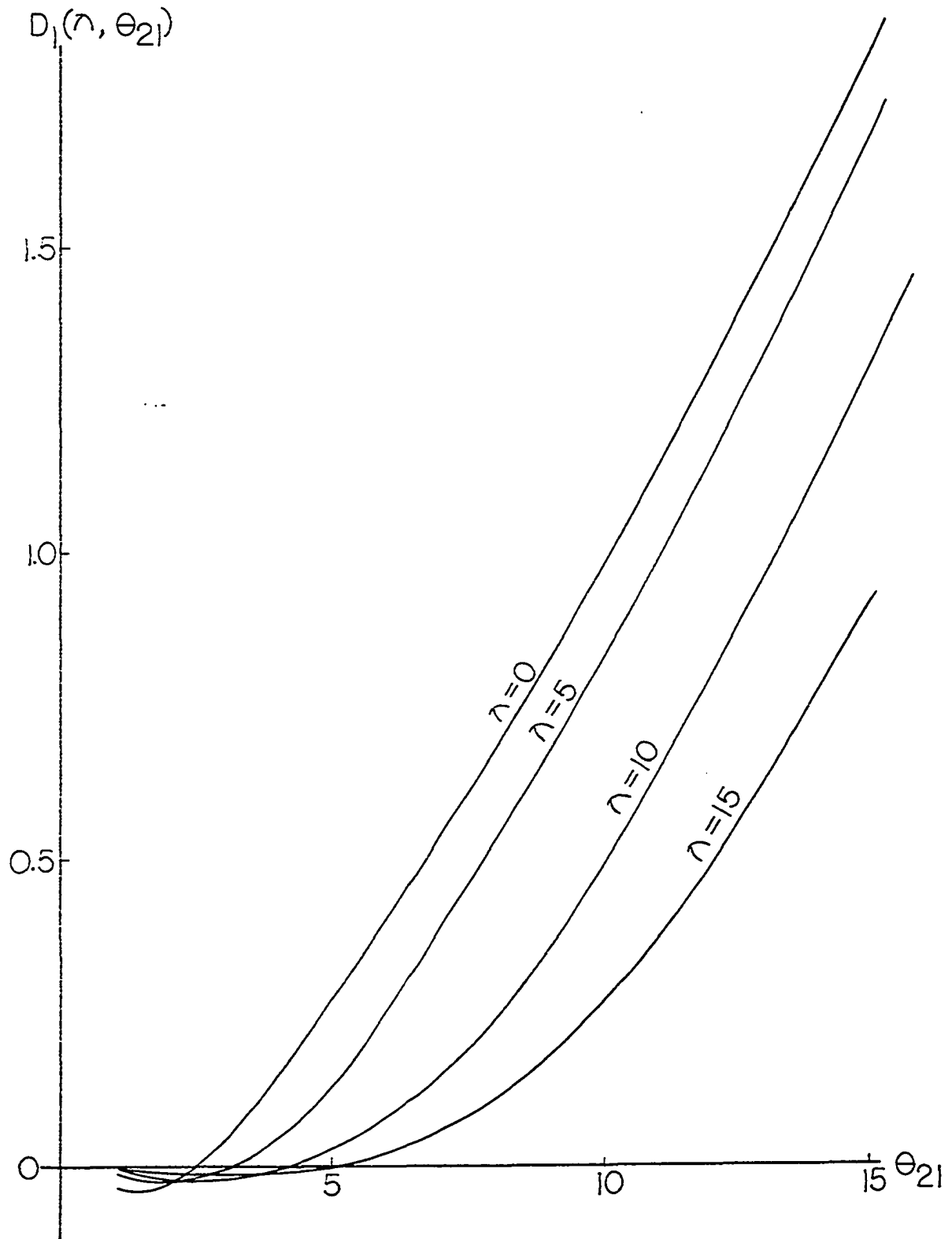


Figure 4.1. Graphs of $D_1(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

We can see that since $\theta_{21} \geq 1$

$$\frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}) > 0 \quad \text{if } \lambda < 1 \text{ or } \lambda > \theta_{21}^2$$

and

$$\frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}) \leq 0 \quad \text{if } 1 \leq \lambda \leq \theta_{21}^2.$$

It follows that for a given $\theta_{21} \geq 1$, $D_1(\lambda, \theta_{21})$ is increasing if $\lambda < 1$ or $\lambda > \theta_{21}^2$ and $D_1(\lambda, \theta_{21})$ is decreasing if $1 < \lambda < \theta_{21}^2$. Further, $D_1(\lambda, \theta_{21})$ reaches its maximum when $\lambda = 1$ and $D_1(\lambda, \theta_{21})$ reaches its minimum when $\lambda = \theta_{21}^2$.

Again, we have

$$\begin{aligned} \frac{\partial^2}{\partial \lambda^2} D_1(\lambda, \theta_{21}) &= \theta_{21}^{\frac{1}{2}f} \lambda^{\frac{1}{2}f - \frac{1}{2}} \left[-(\frac{1}{2}f + \frac{1}{2}) + (\frac{1}{2}f + 1)(1 + \theta_{21})\lambda^{-\frac{1}{2}} - 2\theta_{21}\lambda^{-1} \right. \\ &\quad \left. - (\frac{1}{2}f - 1)\theta_{21}(1 + \theta_{21})\lambda^{-3/2} + \theta_{21}^2(\frac{1}{2}f - 3/2)\lambda^{-2} \right] / \\ &\quad (\theta_{21} + \lambda)^{f+1} B(\frac{1}{2}f, \frac{1}{2}f). \end{aligned} \quad (4.1.12)$$

It can be seen that $\frac{\partial^2}{\partial \lambda^2} D_1(\lambda, \theta_{21})$ tends to zero from below as λ tends to infinity, i.e., when λ is large $D_1(\lambda, \theta_{21})$, as a function of λ , is concave downward. Together with the fact that for any given $\theta_{21} \geq 1$, the horizontal axis is an asymptote of $D_1(\lambda, \theta_{21})$, we can conclude that $D_1(\lambda, \theta_{21})$ must tend to zero from below as λ tends to infinity. Thus the following lemma is proved.

Lemma 4.3 For any given $\theta_{21} \geq 1$,

$$\lim_{\lambda \rightarrow \infty} D_1(\lambda, \theta_{21}) = 0^-.$$

Now we are ready to state and prove a theorem analogues

to Theorem 4.1.

Theorem 4.2 Given $\theta_{21} \geq \theta_m$, where θ_m is defined in 4.1.10, $\exists \lambda_0$ such that

$$D_1(\lambda_0, \theta_{21}) = 0$$

and

$$D_1(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \lambda \leq \lambda_0 .$$

Proof From Corollary 4.1, we know that

$$D_1(0, \theta_{21}) > 0 \quad \text{if } \theta_{21} > \theta_m .$$

In the earlier discussion, we have established that for any given $\theta_{21} \geq 1$, $D_1(\lambda, \theta_{21})$ is increasing if $\lambda < 1$ and is decreasing if $1 < \lambda < \theta_{21}^2$. And Lemma 4.3 tells us that $D_1(\lambda, \theta_{21})$ tends to zero from below as λ tends to infinity. Then $\exists \lambda_0$ such that

$$D_1(\lambda_0, \theta_{21}) = 0$$

and

$$D_1(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \lambda \leq \lambda_0 .$$

This completes the proof of this theorem.

Several graphs are presented in Figure 4.2 to demonstrate the behavior of $D_1(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=0.5$.

B. Comparison of Sometimes Proportional
Allocation with Modified Neyman
Allocation for $k=2$ and $\sigma_1^2 \leq \sigma_2^2$

Now we shall consider the difference function

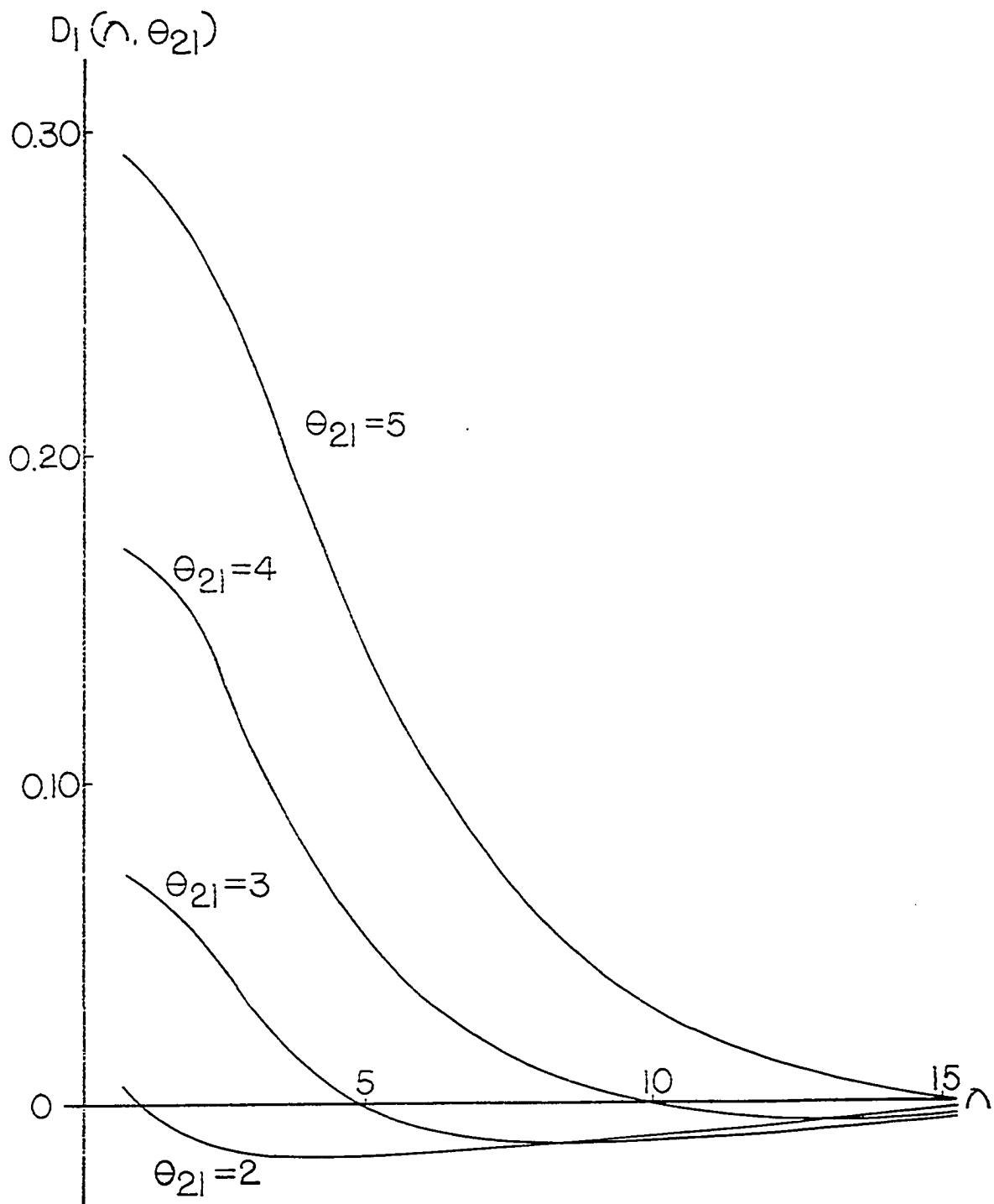


Figure 4.2. Graphs of $D_1(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

$$D_2(\lambda, \theta_{21}) = [V(\bar{y}_w)_N - V(\bar{y}_w)_S] / \frac{w_1 w_2}{n} \sigma_1^2. \quad (4.2.1)$$

Since

$$\begin{aligned} V(\bar{y}_w)_N - V(\bar{y}_w)_S &= V(\bar{y}_w)_P - V(\bar{y}_w)_S - [V(\bar{y}_w)_P - V(\bar{y}_w)_N] \\ &= \frac{w_1 w_2}{n} \sigma_1^2 [D_1(\lambda, \theta_{21}) - D_1(0, \theta_{21})]. \end{aligned} \quad (4.2.2)$$

$D_2(\lambda, \theta_{21})$ can be expressed in terms of $D_1(\lambda, \theta_{21})$, i.e.,

$$D_2(\lambda, \theta_{21}) = D_1(\lambda, \theta_{21}) - D_1(0, \theta_{21}). \quad (4.2.3)$$

Also

$$D_2(\lambda, \theta_{21}) = G \theta_{21}^{\frac{1}{2}} I_{-\frac{1}{2}}(q_{21}) - (1 + \theta_{21}) I_0(q_{21}). \quad (4.2.4)$$

As θ_{21} tends to 1, we have the following lemma concerning $D_2(\lambda, \theta_{21})$.

Lemma 4.4 For any $\lambda \geq 0$, $\lim_{\theta_{21} \rightarrow 1} D_2(\lambda, \theta_{21}) \geq 0$.

Proof From 4.2.3

$$\lim_{\theta_{21} \rightarrow 1} D_2(\lambda, \theta_{21}) = D_2(\lambda, 1) = D_1(\lambda, 1) - D_1(0, 1) \geq 0,$$

since $D_1(\lambda, 1)$ is an increasing function of λ . Q.E.D.

Now let us consider the case that λ is a fixed but arbitrary non-negative number. The following theorem will define a region in which sometimes proportional allocation is more efficient than modified Neyman allocation.

Theorem 4.5 For any $\lambda \geq 0$, $\exists \theta_0 \geq 1$ such that

$$D_2(\lambda, \theta_0) = 0$$

and

$$D_2(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \theta_{21} \leq \theta_0.$$

Proof By Lemma 4.4, there exist some neighborhoods of $\theta_{21}=1$ in which $D_2(\lambda, \theta_{21}) \geq 0$. For any given $\lambda \geq 0$,

$$\begin{aligned} \lim_{\theta_{21} \rightarrow \infty} D_2(\lambda, \theta_{21})/\theta_{21} &= \lim_{\theta_{21} \rightarrow \infty} [D_1(\lambda, \theta_{21}) - D_1(0, \theta_{21})]/\theta_{21} \\ &= \lim_{\theta_{21} \rightarrow \infty} [I_0(p_{21}) - 1] = 0^- , \end{aligned}$$

i.e., $D_2(\lambda, \theta_{21})$ is negative with θ_{21} approaching infinity.

Hence $\exists \theta_0 \geq 1$ such that

$$D_2(\lambda, \theta_0) = 0$$

and

$$D_2(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \theta_{21} \leq \theta_0 .$$

This completes the proof.

For any given non-negative λ , Theorem 4.3 assures us the existence of θ_0 such that for each $\theta_{21} \leq \theta_0$ sometimes proportional allocation is more efficient than modified Neyman allocation.

Again, in Figure 4.3, we present several graphs of $D_2(\lambda, \theta_{21})$.

We shall now consider the case that θ_{21} is a fixed but arbitrary number. As before, we shall prove a theorem analogues to Theorem 4.3 so that for a given θ_{21} there exists λ_0 such that for each $\lambda < \lambda_0$ sometimes proportional allocation is more efficient than modified Neyman allocation. Before we prove the theorem, we shall establish the following lemma first.

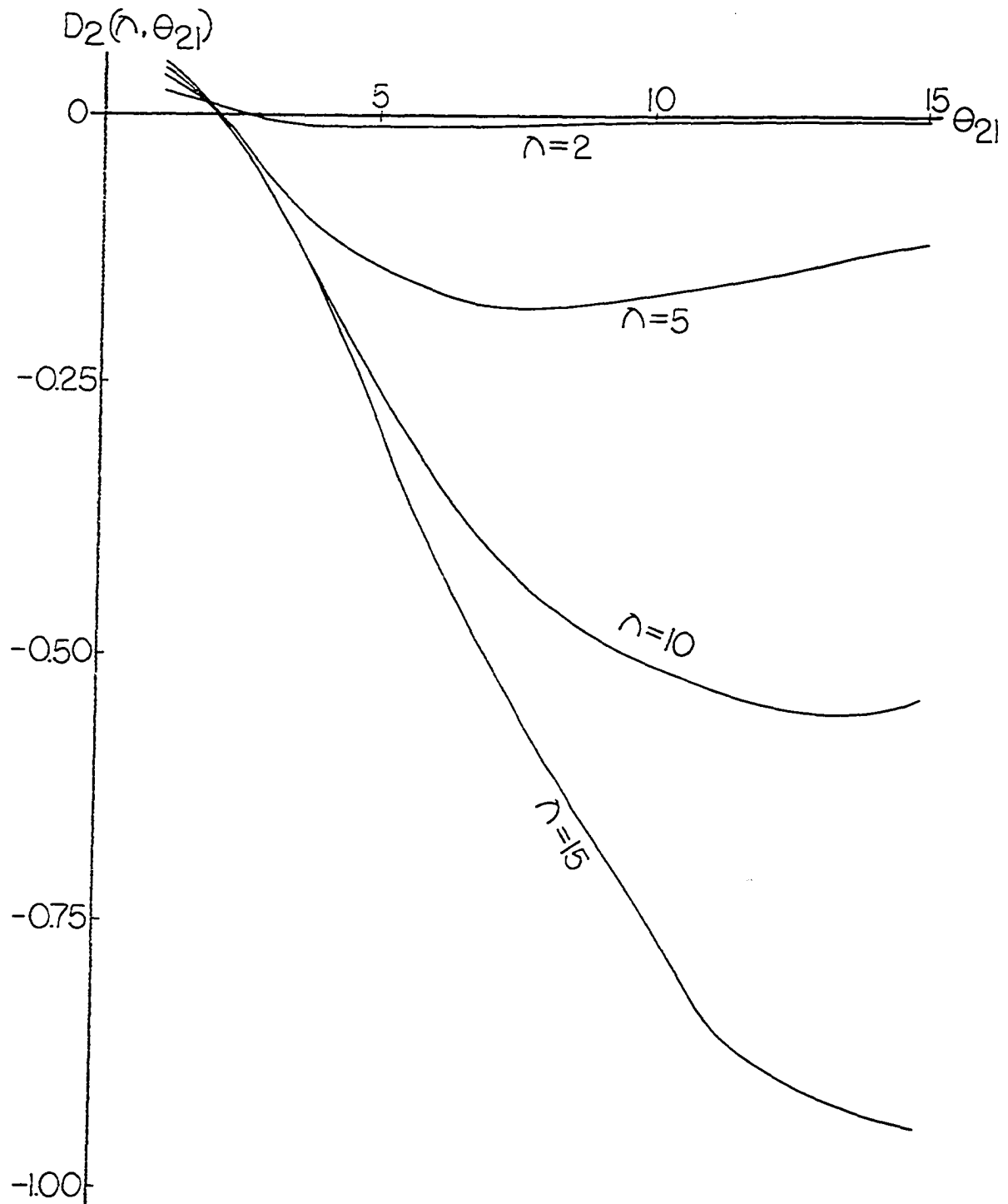


Figure 4.3. Graphs of $D_2(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

Lemma 4.5 For any given $\theta_{21} \geq 1$,

$$\lim_{\lambda \rightarrow 0} D_2(\lambda, \theta_{21}) = 0^+.$$

Proof For any given $\theta_{21} \geq 1$, it can be verified that

$$\lim_{\lambda \rightarrow 0} \theta_{21}^h I_i(q_{21}) = 0^+ \quad \text{for all } h. \quad (4.2.5)$$

From 4.2.4, we therefore have

$$\lim_{\lambda \rightarrow 0} D_2(\lambda, \theta_{21}) = (G-2)0^+ = 0^+.$$

The proof is completed.

Now we are ready to state and prove the following theorem.

Theorem 4.4 For any given $\theta_{21} > \theta_m$ where θ_m is defined in 4.1.10, $\exists \lambda_0$ such that

$$D_2(\lambda_0, \theta_{21}) = 0$$

and

$$D_2(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \lambda \leq \lambda_0.$$

Proof From 4.2.3,

$$\frac{\partial}{\partial \lambda} D_2(\lambda, \theta_{21}) = \frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}).$$

Therefore, for any given $\theta_{21} \geq \theta_m$, $D_2(\lambda, \theta_{21})$ is increasing if $\lambda < 1$ or $\lambda > \theta_{21}^2$. Hence $D_2(1, \theta_{21})$ is the maximum and $D_2(\theta_{21}^2, \theta_{21})$ is the minimum. And from Lemma 4.5, $D_2(0, \theta_{21}) \geq 0$. Thus $D_2(1, \theta_{21})$, the maximum of $D_2(\lambda, \theta_{21})$, is positive.

It can be verified easily that

$$\lim_{\lambda \rightarrow \infty} I_i(q_{21}) = 1. \quad (4.2.6)$$

By using 4.2.6, we obtain

$$\lim_{\lambda \rightarrow \infty} D_2(\lambda, \theta_{21}) = -(\theta_{21} - G\theta_{21}^{\frac{1}{2}} + 1) < 0.$$

Since $D_2(\lambda, \theta_{21})$ is increasing if $\lambda > \theta_{21}^2$ and $D_2(\lambda, \theta_{21})$ tends to a negative limit as λ tends to infinity, $\exists \lambda_0$ such that

$$D_2(\lambda_0, \theta_{21}) = 0$$

and

$$D_2(\lambda, \theta_{21}) \geq 0 \quad \text{for each } \lambda \leq \lambda_0.$$

That completes the proof of the theorem.

Several graphs of $D_2(\lambda, \theta_{21})$ are presented in Figure 4.4.

C. Comparison of Sometimes Proportional

Allocation with Proportional Allocation

for $k=2$ and $\sigma_1^2 \neq \sigma_2^2$

Now consider the difference function

$$D_1^*(\lambda, \theta_{21}) = [V(\bar{y}_w)_P - V(\bar{y}_w)_S] / \frac{w_1 w_2}{n} \sigma_1^2, \quad (4.3.1)$$

where $V(\bar{y}_w)_S$ is given in 3.2.14. Then 4.3.1 can be written explicitly as

$$D_1^*(\lambda, \theta_{21}) = (1 + \theta_{21}) [I_0(p_{21}) + I_0(q_{21}^{(1)})] - G\theta_{21}^{\frac{1}{2}} [I_{-\frac{1}{2}}(p_{21}) + I_{-\frac{1}{2}}(q_{21}^{(1)})], \quad (4.3.2)$$

where

$$I_i(x) = I_x(\frac{1}{2}f+i, \frac{1}{2}f+i), \quad (4.3.3)$$

and

$$q_{21}^{(1)} = 1 - p_{21}^{(1)} = \frac{1}{\lambda \theta_{21} + 1}. \quad (4.3.4)$$

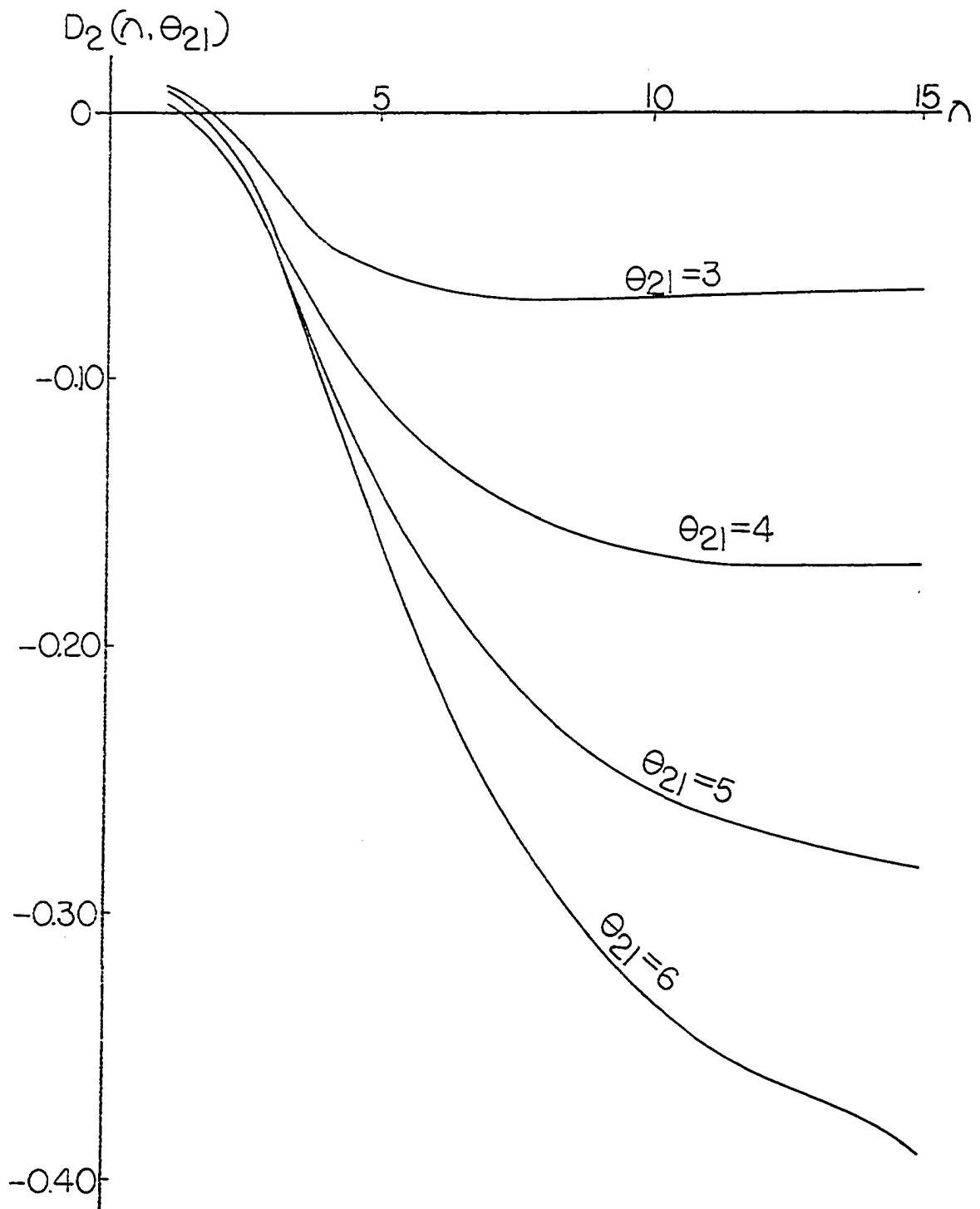


Figure 4.4. Graphs of $D_2(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

Now, it can be easily verified that

$$D_1^*(\lambda, \theta_{21}) = D_1(\lambda, \theta_{21}) - D_2(1/\lambda, \theta_{21}),$$

whence, using 4.2.3, we obtain

$$D_1^*(\lambda, \theta_{21}) = D_1(\lambda, \theta_{21}) - D_1(1/\lambda, \theta_{21}) + (1 + \theta_{21} - G\theta_{21}^{\frac{1}{2}}). \quad (4.3.5)$$

As θ_{21} tends to 1, we have the following lemma for $D_1^*(\lambda, \theta_{21})$.

Lemma 4.6 For any $\lambda \geq 1$, $\lim_{\theta_{21} \rightarrow 1} D_1^*(\lambda, \theta_{21}) \leq 0$.

Proof For any $\lambda \geq 1$, by Lemma 4.1

$$\lim_{\theta_{21} \rightarrow 1} D_1(\lambda, \theta_{21}) \leq 0, \text{ and by Lemma 4.4 } \lim_{\theta_{21} \rightarrow 1} D_2(\lambda, \theta_{21}) \geq 0.$$

From 4.3.5, we have

$$\lim_{\theta_{21} \rightarrow 1} D_1^*(\lambda, \theta_{21}) = \lim_{\theta_{21} \rightarrow 1} D_1(\lambda, \theta_{21}) - \lim_{\theta_{21} \rightarrow 1} D_2(1/\lambda, \theta_{21}) \leq 0.$$

Q.E.D.

Next let us consider the case that λ is a fixed but arbitrary number. We shall discuss the behavior of $D_1^*(\lambda, \theta_{21})$ as θ_{21} varies by first proving the following lemmas.

Lemma 4.7 For any given $\lambda \geq 1$,

$$\frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) < 0 \quad \text{for } 0 < \theta_{21} < 1,$$

Proof From 4.3.2, taking partial derivative of

$D_1^*(\lambda, \theta_{21})$ with respect to θ_{21} , we have

$$\begin{aligned} \frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) &= [I_0(p_{21}) + I_0(q_{21}^{(1)})] - \frac{1}{2} G \theta_{21}^{-\frac{1}{2}} [I_{-\frac{1}{2}}(p_{21}) + I_{-\frac{1}{2}}(q_{21}^{(1)})] \\ &\quad + (1 + \theta_{21}) [R_0(p_{21}) + R_0(q_{21}^{(1)})] - G \theta_{21}^{\frac{1}{2}} [R_{-\frac{1}{2}}(p_{21}) + R_{-\frac{1}{2}}(q_{21}^{(1)})], \end{aligned} \quad (4.3.6)$$

where $R_i(p_{21})$ is defined in 4.1.6

and

$$R_i(q_{21}^{(1)}) = \frac{\partial}{\partial \theta_{21}} I_i(q_{21}^{(1)}) = -p_{21}^{(1)\frac{1}{2}f+i} q_{21}^{(1)\frac{1}{2}f+i} / \theta_{21}^{B(\frac{1}{2}f+i, \frac{1}{2}f+i)}. \quad (4.3.7)$$

Substituting 4.3.7 in 4.3.6, we obtain, after simplification,

$$\begin{aligned} & \frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) \\ &= [I_0(p_{21}) - \frac{1}{2} G \theta_{21}^{-\frac{1}{2}} I_{-\frac{1}{2}}(p_{21})] \div [I_0(q_{21}^{(1)}) - \frac{1}{2} G \theta_{21}^{-\frac{1}{2}} I_{-\frac{1}{2}}(q_{21}^{(1)})] \\ &+ \lambda^{\frac{1}{2}f - \frac{1}{2}} \theta_{21}^{\frac{1}{2}f - 1} \left\{ (1 + \theta_{21})^{\lambda^{\frac{1}{2}}} [(\lambda + \theta_{21})^{-f} - (\lambda \theta_{21} + 1)^{-f}] \right. \\ &\quad \left. - [(\lambda + \theta_{21})^{-f+1} - (\lambda \theta_{21} + 1)^{-f+1}] \right\} / B(\frac{1}{2}f, \frac{1}{2}f). \end{aligned} \quad (4.3.8)$$

It can be verified that for any x such that $0 < x < 1$,

$$I_0(x) - \frac{1}{2} G \theta_{21}^{-\frac{1}{2}} I_{-\frac{1}{2}}(x) < 0. \quad (4.3.9)$$

Thus, from 4.3.9, the first two terms in 4.3.8 are negative.

Now let us examine the third term in 4.3.8. The third term can be written as

$$\frac{\lambda^{\frac{1}{2}f - \frac{1}{2}} \theta_{21}^{\frac{1}{2}f - 1} (\lambda^{\frac{1}{2}} - 1) [(\lambda \theta_{21} + 1)^f (\theta_{21} - \lambda^{\frac{1}{2}}) + (\lambda + \theta_{21})^f (\lambda^{\frac{1}{2}} \theta_{21} - 1)]}{(\lambda + \theta_{21})^f (\lambda \theta_{21} + 1)^f B(\frac{1}{2}f, \frac{1}{2}f)}. \quad (4.3.10)$$

Since

$$(\lambda \theta_{21} + 1)^f (\theta_{21} - \lambda^{\frac{1}{2}}) + (\lambda + \theta_{21})^f (\lambda^{\frac{1}{2}} \theta_{21} - 1) < (\lambda + 1)^f (\lambda^{\frac{1}{2}} + 1) (\theta_{21} - 1) < 0$$

then the third term in 4.3.8 is also negative.

Hence, for $0 < \theta_{21} < 1$, $\frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) < 0$. The proof is completed.

Lemma 4.8 For any given $\lambda \geq 1$ satisfying

$$1 - \frac{1}{2}G + (\lambda^{\frac{1}{2}f - \frac{1}{2}} + 1)(\lambda^{\frac{1}{2}} - 1) / \lambda^{\frac{1}{2}f} B(\frac{1}{2}f, \frac{1}{2}f) > 0, \quad (4.3.11)$$

$\exists \theta' > 1$ such that

$$\frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) > 0 \quad \forall \quad \theta_{21} > \theta'$$

Proof From 4.3.8, $\frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21})$ can also be

written as

$$\begin{aligned} \frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) &= [\bar{I}_0(p_{21}) + \bar{I}_0(q_{21}^{(1)})] - \frac{1}{2}G\theta_{21}^{-\frac{1}{2}} [\bar{I}_{-\frac{1}{2}}(p_{21}) + \bar{I}_{-\frac{1}{2}}(q_{21}^{(1)})] \\ &\quad + \lambda^{\frac{1}{2}f - \frac{1}{2}} \theta_{21}^{\frac{1}{2}f - 1} \left\{ (1 + \theta_{21}) \lambda^{\frac{1}{2}} [(\lambda + \theta_{21})^{-f} - (\lambda \theta_{21} + 1)^{-f}] \right. \\ &\quad \left. - [(\lambda + \theta_{21})^{-f+1} - (\lambda \theta_{21} + 1)^{-f+1}] \right\} / B(\frac{1}{2}f, \frac{1}{2}f). \end{aligned} \quad (4.3.12)$$

It can be verified that, for $h \geq 0$,

$$\lim_{\theta_{21} \rightarrow \infty} [\bar{I}_h(p_{21}) + \bar{I}_h(q_{21}^{(1)})] / \theta_{21}^h = 0^+. \quad (4.3.13)$$

Using 4.3.13, we find that

$$\lim_{\theta_{21} \rightarrow \infty} \frac{1}{\theta_{21}} \frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) = [1 - \frac{1}{2}G + (\lambda^{\frac{1}{2}f - \frac{1}{2}} + 1)(\lambda^{\frac{1}{2}} - 1) / \lambda^{\frac{1}{2}f} B(\frac{1}{2}f, \frac{1}{2}f)] 0^+ = 0^+.$$

This implies that $\exists \theta'$ such that

$$\frac{\partial}{\partial \theta_{21}} D_1^*(\lambda, \theta_{21}) > 0 \quad \forall \quad \theta_{21} > \theta'.$$

The proof of the lemma is completed.

Lemmas 4.7 and 4.8 tell us that, for a given λ , $D_1^*(\lambda, \theta_{21})$, as a function of θ_{21} , decreases first and then increases. After the above three lemmas are established, we are ready to prove the following theorem.

Theorem 4.5 For any given $\lambda \geq 1$, satisfying 4.3.11, $\exists \theta_0^{(1)}$ in $(0,1)$ and $\theta_0^{(2)} > 1$ such that

$$D_1^*(\lambda, \theta_0^{(1)}) = D_1^*(\lambda, \theta_0^{(2)}) = 0$$

and

$$D_1^*(\lambda, \theta_{21}) > 0 \quad \forall \quad \theta_{21} < \theta_0^{(1)} \quad \text{or} \quad \theta_{21} > \theta_0^{(2)}.$$

Proof Because of lemmas 4.6, 4.7 and 4.8, it suffices to prove this theorem by showing that $D_1^*(\lambda, 0) > 0$ and $\lim_{\theta_{21} \rightarrow \infty} D_1^*(\lambda, \theta_{21}) > 0$.

First, consider $D_1^*(\lambda, 0)$. When $\theta_{21}=0$, $p_{21}=0$ and $q_{21}^{(1)}=1$. Thus $D_1^*(\lambda, 0)=1 > 0$.

It can be seen that

$$\lim_{\theta_{21} \rightarrow \infty} D_1^*(\lambda, \theta_{21}) / \theta_{21} = 1.$$

This implies that $\lim_{\theta_{21} \rightarrow \infty} D_1^*(\lambda, \theta_{21}) > 0$.

The existence of $\theta_0^{(1)}$ and $\theta_0^{(2)}$ is in evidence. The proof is completed.

For any given $\lambda \geq 1$, satisfying 4.3.11, Theorem 4.5 assures us that there exist $\theta_0^{(1)}$ in $(0,1)$ and $\theta_0^{(2)} > 1$ such that for each $\theta_{21} < \theta_0^{(1)}$ or $\theta_{21} > \theta_0^{(2)}$ sometimes proportional allocation is always more efficient than proportional allocation.

Taking, in particular, $\lambda=1$, we obtain the following corollary to Theorem 4.5.

Corollary 4.2

$$D_1^*(1, \theta_{21}) > 0 \quad \text{if } \theta_{21} < \theta_m^{(1)} \quad \text{or } \theta_{21} > \theta_m^{(2)}$$

and

$$D_1^*(1, \theta_{21}) \leq 0 \quad \text{if } \theta_m^{(1)} \leq \theta_{21} \leq \theta_m^{(2)},$$

where

$$\theta_m^{(1)} = \frac{1}{2}(G^2 - 2 - G\sqrt{G^2 - 4}),$$

and

$$\theta_m^{(2)} = \frac{1}{2}(G^2 - 2 + G\sqrt{G^2 - 4}). \quad (4.3.14)$$

Proof When $\lambda=1$, $p_{21} = \theta_{21}/(1+\theta_{21})$ and

$$q_{21}^{(1)} = 1/(1+\theta_{21}) = q_{21}. \quad \text{Hence}$$

$$I_i(p_{21}) + I_i(q_{21}^{(1)}) = 1. \quad (4.3.15)$$

Using 4.3.15 or from 4.3.5, $D_1^*(1, \theta_{21}) = \theta_{21} - G\theta_{21}^{\frac{1}{2}} + 1$ which is a quadratic function in $\theta_{21}^{\frac{1}{2}}$. Therefore $D_1^*(1, \theta_{21})$ has two

zeros $\theta_m^{(1)}$ and $\theta_m^{(2)}$ defined in 4.3.14. Thus $D_1^*(1, \theta_{21}) > 0$

for $\theta_{21} < \theta_m^{(1)}$ or $\theta_{21} > \theta_m^{(2)}$ and $D_1^*(1, \theta_{21}) \leq 0$ for

$$\theta_m^{(1)} \leq \theta_{21} \leq \theta_m^{(2)}. \quad \text{Q.E.D.}$$

If the set S is defined as

$$S = \{\theta_{21} : \theta_m^{(1)} \leq \theta_{21} \leq \theta_m^{(2)}\}, \quad (4.3.16)$$

where $\theta_m^{(1)}$ and $\theta_m^{(2)}$ are given in 4.3.14, and S' is the complementary set of S , then $D_1^*(1, \theta_{21}) > 0$ if θ_{21} is in S' and $D_1^*(1, \theta_{21}) \leq 0$ if θ_{21} is in S .

We shall now consider the case when θ_{21} is a fixed but

arbitrary number. The behavior of $D_1^*(\lambda, \theta_{21})$, as λ varies, will be discussed in the following lemma and theorem.

Lemma 4.9 For any given $\theta_{21} \geq 0$, $\exists \lambda'$ such that $\frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21}) > 0 \quad \forall \lambda > \lambda'$.

Proof From 4.3.2, taking partial derivative with respect to λ , we have

$$\frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21}) = (1 + \theta_{21}) [Q_0(p_{21}) + Q_0(q_{21}^{(1)})] - G\theta_{21}^{\frac{1}{2}} [Q_{-\frac{1}{2}}(p_{21}) + Q_{-\frac{1}{2}}(q_{21}^{(1)})], \quad (4.3.17)$$

Where $Q_i(p_{21})$ is defined in 4.1.5

$$\text{and} \quad Q_i(q_{21}^{(1)}) = \frac{\partial}{\partial \lambda} I_i(q_{21}^{(1)}) = -p_{21}^{(1)\frac{1}{2}f+i} q_{21}^{(1)\frac{1}{2}f+i} / \lambda B(\frac{1}{2}f+i, \frac{1}{2}f+i). \quad (4.3.18)$$

Substituting 4.3.18 in 4.3.17, we obtain, after simplification

$$\frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21}) = \theta_{21}^{\frac{1}{2}f} \lambda^{\frac{1}{2}f-3/2} (\lambda^{\frac{1}{2}} - 1) \left[\frac{\lambda^{\frac{1}{2}-\theta_{21}}}{(\lambda + \theta_{21})^f} + \frac{\lambda^{\frac{1}{2}\theta_{21}-1}}{(\lambda\theta_{21} + 1)^f} \right] / B(\frac{1}{2}f, \frac{1}{2}f). \quad (4.3.19)$$

$$\text{Then } \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21}) = [\theta_{21}^{\frac{1}{2}f} (1 + \theta_{21}^{-f+1}) / B(\frac{1}{2}f, \frac{1}{2}f)] 0^+ = 0^+.$$

Hence, $\exists \lambda'$ such that

$$\frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21}) > 0 \quad \forall \lambda > \lambda'.$$

The proof of Lemma 4.9 is completed.

If we consider the limit of $\frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21})$, as λ tends to infinity, we get

$$\lim_{\lambda \rightarrow \infty} \frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21}) = 0^+,$$

i.e., $D_1^*(\lambda, \theta_{21})$ has a horizontal asymptote and $D_1^*(\lambda, \theta_{21})$ tends to its horizontal asymptote from below. Noting that $\lim_{\lambda \rightarrow \infty} D_1^*(\lambda, \theta_{21}) = 0$, we see that $D_1^*(\lambda, \theta_{21})$ tends to zero from below. Thus for any given θ_{21} in S' , where S is defined in 4.3.16, there exists λ_0 such that $D_1^*(\lambda_0, \theta_{21}) = 0$ and $D_1^*(\lambda, \theta_{21}) \geq 0$ for each $\lambda \leq \lambda_0$. The following theorem is then proved.

Theorem 4.6 For any given θ_{21} in S' , $\exists \lambda_0$ such that

$$D_1^*(\lambda_0, \theta_{21}) = 0$$

and

$$D_1^*(\lambda, \theta_{21}) \geq 0 \quad \forall \lambda \leq \lambda_0.$$

For any given θ_{21} in S' , Theorem 4.6 assures us that there exists λ_0 such that sometimes proportional allocation will be always more efficient than proportional allocation for each $\lambda < \lambda_0$.

D. Comparison of Sometimes Proportional Allocation with Modified Neyman Allocation for $k=2$ and $\sigma_1^2 \neq \sigma_2^2$

Now we shall consider the difference function

$$D_2^*(\lambda, \theta_{21}) = [V(\bar{y}_w)_{\underline{N}} - V(\bar{y}_w)_S] / \frac{w_1 w_2}{n} \sigma_1^2, \quad (4.4.1)$$

From the identity,

$$V(\bar{y}_w)_{\underline{N}} - V(\bar{y}_w)_S = V(\bar{y}_w)_P - V(\bar{y}_w)_S - [V(\bar{y}_w)_P - V(\bar{y}_w)_{\underline{N}}],$$

$D_2^*(\lambda, \theta_{21})$ can be expressed in terms of $D_1^*(\lambda, \theta_{21})$, i.e.,

$$\begin{aligned} D_2^*(\lambda, \theta_{21}) &= D_1^*(\lambda, \theta_{21}) - (\theta_{21} - G\theta_{21}^{\frac{1}{2}} + 1) \\ &= D_1^*(\lambda, \theta_{21}) - D_1^*(1, \theta_{21}). \end{aligned} \quad (4.4.2)$$

Using 4.3.5 $D_2^*(\lambda, \theta_{21})$ can also be expressed in terms of $D_1(\lambda, \theta_{21})$, i.e.,

$$D_2^*(\lambda, \theta_{21}) = D_1(\lambda, \theta_{21}) - D_1(1/\lambda, \theta_{21}). \quad (4.4.3)$$

And, further, more explicitly,

$$D_2^*(\lambda, \theta_{21}) = G\theta_{21}^{\frac{1}{2}} [\bar{I}_{-\frac{1}{2}}(p_{21}^{(1)}) - I_{-\frac{1}{2}}(p_{21})] - (1 + \theta_{21}) [I_0(p_{21}^{(1)}) - I_0(p_{21})], \quad (4.4.4)$$

where $p_{21}^{(1)}$ is defined in 4.3.4.

As θ_{21} tends to 1, we have the following lemma for $D_2^*(\lambda, \theta_{21})$.

Lemma 4.10 For any given $\lambda \geq 1$,

$$\lim_{\theta_{21} \rightarrow 1} D_2^*(\lambda, \theta_{21}) \geq 0.$$

Proof From 4.4.2, we have

$$\lim_{\theta_{21} \rightarrow 1} D_2^*(\lambda, \theta_{21}) = D_2^*(\lambda, 1) = D_1^*(\lambda, 1) - D_1^*(1, 1).$$

Letting $\theta_{21} = 1$ in 4.3.17, we obtain the partial derivative of $D_1^*(\lambda, 1)$ with respect to λ , namely

$$\begin{aligned} \frac{\partial}{\partial \lambda} D_1^*(\lambda, 1) &= 2 \left[2Q_0\left(\frac{1}{\lambda+1}\right) - GQ_{-\frac{1}{2}}\left(\frac{1}{\lambda+1}\right) \right] \\ &= 2 \left[-2\left(\frac{1}{\lambda+1}\right)^{\frac{1}{2}f} \left(\frac{\lambda}{\lambda+1}\right)^{\frac{1}{2}f} + \left(\frac{1}{\lambda+1}\right)^{\frac{1}{2}f-\frac{1}{2}} \left(\frac{\lambda}{\lambda+1}\right)^{\frac{1}{2}f-\frac{1}{2}} \right] / \lambda B\left(\frac{1}{2}f, \frac{1}{2}f\right) \\ &= 2\lambda^{\frac{1}{2}f-5/2} (\lambda^{\frac{1}{2}} - 1)^2 / (\lambda+1)^f B\left(\frac{1}{2}f, \frac{1}{2}f\right) \geq 0. \end{aligned} \quad (4.4.5)$$

From 4.4.5 we conclude that $D_1^*(\lambda, 1)$ is a non-decreasing function of λ . Thus $D_1^*(\lambda, 1) - D_1^*(1, 1) \geq 0$. Therefore,

$$\lim_{\theta_{21} \rightarrow 1} D_2^*(\lambda, \theta_{21}) \geq 0. \quad \text{The proof is complete.}$$

Next let us consider the case when λ is a fixed but arbitrary number. We shall discuss the behavior of $D_2^*(\lambda, \theta_{21})$ as θ_{21} varies.

It can be verified that

$$\lim_{\theta_{21} \rightarrow \infty} [I_i(p_{21}^{(1)}) - I_i(p_{21})] = 0^+. \quad (4.4.6)$$

From 4.4.4, we have, on using the above result and Corollary

$$4.1 \quad \lim_{\theta_{21} \rightarrow \infty} D_2^*(\lambda, \theta_{21}) / \theta_{21} = \lim_{\theta_{21} \rightarrow \infty} \left[\frac{G}{\theta_{21}^2} - \left(\frac{1}{\theta_{21}} + 1 \right) \right] 0^+ = 0^-.$$

This implies that

$$\lim_{\theta_{21} \rightarrow \infty} D_2^*(\lambda, \theta_{21}) \leq 0. \quad (4.4.7)$$

It follows from Lemma 4.10 that there exists $\theta_0^{(2)} > 1$ such that

$$D_2^*(\lambda, \theta_0^{(2)}) = 0$$

and

$$D_2^*(\lambda, \theta_{21}) \geq 0 \quad \forall \quad \theta_{21} \text{ in } [1, \theta_0^{(2)}].$$

Furthermore, as θ_{21} tends to 0,

$$\lim_{\theta_{21} \rightarrow 0} D_2^*(\lambda, \theta_{21}) = 0^-.$$

Hence, $\exists \theta_0^{(1)}$ in $(0, 1)$ such that

$$D_2^*(\lambda, \theta_0^{(1)}) = 0$$

and

$$D_2^*(\lambda, \theta_{21}) \geq 0 \quad \forall \theta_{21} \text{ in } [\theta_0^{(1)}, 1].$$

Thus the following theorem is proved.

Theorem 4.7 For any given $\lambda \geq 1$, $\exists \theta_0^{(1)}$ in $(0, 1)$ and $\theta_0^{(2)} > 1$ such that

$$D_2^*(\lambda, \theta_0^{(1)}) = D_2^*(\lambda, \theta_0^{(2)}) = 0$$

and

$$D_2^*(\lambda, \theta_{21}) \geq 0 \quad \forall \theta_{21} \text{ in } [\theta_0^{(1)}, \theta_0^{(2)}].$$

For any given $\lambda \geq 1$, Theorem 4.7 assures us that there exist $\theta_0^{(1)}$ in $(0, 1)$ and $\theta_0^{(2)} > 1$ such that for each θ_{21} in $(\theta_0^{(1)}, \theta_0^{(2)})$ sometimes proportional allocation is always more efficient than modified Neyman allocation.

We shall now consider the case when θ_{21} is a fixed but arbitrary number. As λ tends to 1, we have the following lemma.

Lemma 4.11 For any given $\theta_{21} \geq 0$,

$$\lim_{\lambda \rightarrow 1} D_2^*(\lambda, \theta_{21}) = 0.$$

Proof From 4.4.2, we have

$$\lim_{\lambda \rightarrow 1} D_2^*(\lambda, \theta_{21}) = \lim_{\lambda \rightarrow 1} D_1^*(\lambda, \theta_{21}) - D_1^*(1, \theta_{21}) = 0. \quad \text{Q.E.D.}$$

Lemma 4.12 For any given θ_{21} in S , where S is defined in 4.3.16, $\lim_{\lambda \rightarrow \infty} D_2^*(\lambda, \theta_{21}) \geq 0$.

Proof From 4.4.2, we have

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} D_2^*(\lambda, \theta_{21}) &= \lim_{\lambda \rightarrow \infty} D_1^*(\lambda, \theta_{21}) - (\theta_{21} - G\theta_{21}^{\frac{1}{2}} + 1) \\
&= -(\theta_{21} - G\theta_{21}^{\frac{1}{2}} + 1) \geq 0. \quad \text{Q.E.D.}
\end{aligned}$$

After the above two lemmas are established, we are now ready to prove the following theorem.

Theorem 4.8 For any given θ_{21} in S , where S is defined in 4.3.16, $\exists \lambda_0$ such that

$$D_2^*(\lambda_0, \theta_{21}) = 0$$

and

$$D_2^*(\lambda, \theta_{21}) \geq 0 \quad \forall \lambda \geq \lambda_0.$$

Proof From 4.4.2, we have $\frac{\partial}{\partial \lambda} D_2^*(\lambda, \theta_{21}) = \frac{\partial}{\partial \lambda} D_1^*(\lambda, \theta_{21})$.

By Lemma 4.9, $\exists \lambda'$ such that $\frac{\partial}{\partial \lambda} D_2^*(\lambda, \theta_{21}) > 0 \quad \forall \lambda > \lambda'$, i.e., $D_2^*(\lambda, \theta_{21})$ is increasing for each $\lambda > \lambda'$. We shall consider the following two cases:

Case 1 For any given θ_{21} in S , if $D_2^*(\lambda, \theta_{21}) \geq 0$ for $\forall \lambda \geq 1$, we can let $\lambda_0 = 1$. Thus

$$D_2^*(\lambda_0, \theta_{21}) = 0$$

and

$$D_2^*(\lambda, \theta_{21}) \geq 0 \quad \forall \lambda \geq \lambda_0.$$

Case 2 For any given θ_{21} in S , if $D_2^*(\lambda, \theta_{21}) < 0$ for some λ , because of Lemma 4.11, $\exists \lambda_0$ such that

$$D_2^*(\lambda_0, \theta_{21}) = 0$$

and

$$D_2^*(\lambda, \theta_{21}) \geq 0 \quad \forall \lambda \geq \lambda_0.$$

In either case the existence of λ_0 is in evidence. The proof of Theorem 4.8 is completed.

For any given θ_{21} in S, Theorem 4.8 assures us that there exists λ_0 such that sometimes proportional allocation will be more efficient than modified Neyman allocation for each $\lambda > \lambda_0$.

E. Comparison of Sometimes Proportional
Allocation with Proportional Allocation
for $k=3$ and $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2$

Consider the difference function

$$\begin{aligned}
 D_1(\lambda, \theta_{21}, \theta_{32}) &= [V(\bar{y}_w)_P - V(\bar{y}_w)_S] / \frac{\sigma_2^2}{n} \\
 &= [w_1(1-w_1)/\theta_{21} + w_2(1-w_2) + w_3(1-w_3)\theta_{32}] P(A_1'') \\
 &\quad - G[w_1 w_2 I_{p_{21}}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f - \frac{1}{2}) / \theta_{21}^{\frac{1}{2}} + w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} + w_2 w_3 \theta_{32}^{\frac{1}{2}}] \\
 &\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} A(\lambda, \theta_{21}, \theta_{32}) \\
 &\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} B(\lambda, \theta_{21}, \theta_{32}), \quad (4.5.1)
 \end{aligned}$$

where $P(A_1'')$, $A(\lambda, \theta_{21}, \theta_{32})$ and $B(\lambda, \theta_{21}, \theta_{32})$ are defined in 3.3.39, 3.3.41 and 3.3.42 respectively.

Clearly, if $D_1(\lambda, \theta_{21}, \theta_{32}) > 0$ sometimes proportional allocation is more efficient than proportional allocation. Otherwise, proportional allocation is more efficient.

If we let

$$I_{i,j}(x) = I_x(\frac{1}{2}f+i, f-1+j-r), \quad (4.5.2)$$

and

$$I_{i,j}^c(x) = I_x(f-1+j-r, \frac{1}{2}f+i), \quad (4.5.3)$$

then, using 4.5.2 and 4.5.3, $P(A_0'')$, $A(\lambda, \theta_{21}, \theta_{32})$ and

$B(\lambda, \theta_{21}, \theta_{32})$ can be written as follows.

$$P(A_0'') = \sum_{r=0}^{\frac{1}{2}f-1} \binom{f-2-r}{\frac{1}{2}f-1-r} [q_{32}^{\frac{1}{2}f} p_{32}^{\frac{1}{2}f-1-r} I_{0,0}^c(p_{32}^*) - p_{21}^{\frac{1}{2}f} q_{21}^{\frac{1}{2}f-1-r} I_{0,0}(q_{31}^*)], \quad (4.5.4)$$

$$\begin{aligned} A(\lambda, \theta_{21}, \theta_{32}) &= (w_1 w_2 p_{31}^{\frac{1}{2}f-\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f-\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r}) \cdot \\ &\quad [I_{\frac{1}{2}, -\frac{1}{2}}(\bar{q}_{21}) - I_{\frac{1}{2}, -\frac{1}{2}}(q_{21}^*)] \\ &\quad + (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f-\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f-\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r}) I_{\frac{1}{2}, -\frac{1}{2}}(q_{31}^*) \\ &\quad + (w_1 w_2 p_{32}^{\frac{1}{2}f-\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f-\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}}) I_{\frac{1}{2}, -\frac{1}{2}}^c(p_{32}^*), \end{aligned} \quad (4.5.5)$$

and

$$\begin{aligned} B(\lambda, \theta_{21}, \theta_{32}) &= (w_1 w_2 p_{31}^{\frac{1}{2}f+\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f+\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r}) \cdot \\ &\quad [I_{-\frac{1}{2}, \frac{1}{2}}(\bar{q}_{21}) - I_{-\frac{1}{2}, \frac{1}{2}}(q_{21}^*)] \\ &\quad + (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f+\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f+\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r}) I_{-\frac{1}{2}, \frac{1}{2}}(q_{31}^*) \\ &\quad + (w_1 w_2 p_{32}^{\frac{1}{2}f+\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f+\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}}) I_{-\frac{1}{2}, \frac{1}{2}}^c(p_{32}^*). \end{aligned} \quad (4.5.6)$$

Let us consider the case that λ is a fixed but arbitrary non-negative number. We shall discuss the behavior of $D_1(\lambda, \theta_{21}, \theta_{32})$ as θ_{21} and θ_{32} vary. Let $R_{i,j}(x)$ and $R_{i,j}^c(x)$ denote partial derivatives of $I_{i,j}(x)$ and $I_{i,j}^c(x)$ with respect to θ_{21} respectively, where $I_{i,j}(x)$ and $I_{i,j}^c(x)$ are defined in

4.5.2 and 4.5.3. Then

$$\begin{aligned}
 R_{i,j}(\bar{q}_{21}) &= \frac{\partial}{\partial \theta_{21}} I_{i,j}(\bar{q}_{21}) \\
 &= \frac{\partial}{\partial \theta_{21}} I_{\bar{q}_{21}}(\tfrac{1}{2}f+i, f-1+j-r) \\
 &= \frac{\partial}{\partial \theta_{21}} \int_0^{\bar{q}_{21}} x^{\frac{1}{2}f+i-1} (1-x)^{f-2+j-r} dx / B(\tfrac{1}{2}f+i, f-1+j-r) \\
 &= -\frac{\frac{1}{2}f+i-1}{\bar{q}_{21}^{\frac{1}{2}f+i}} \bar{p}_{21}^{f-2+j-r} / \lambda B(\tfrac{1}{2}f+i, f-1+j-r) . \quad (4.5.7)
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 R_{i,j}(q_{21}^*) &= \frac{\partial}{\partial \theta_{21}} I_{i,j}(q_{21}^*) \\
 &= -q_{21}^{*\frac{1}{2}f+i+1} p_{21}^{*f-2+j-r} / B(\tfrac{1}{2}f+i, f-1+j-r) , \quad (4.5.8)
 \end{aligned}$$

$$\begin{aligned}
 R_{i,j}^c(p_{32}^*) &= \frac{\partial}{\partial \theta_{21}} I_{i,j}^c(p_{32}^*) \\
 &= -p_{32}^{*f-1+j-r} q_{32}^{*\frac{1}{2}f+i} / \theta_{21} B(f-1+j-r, \tfrac{1}{2}f+i) , \quad (4.5.9)
 \end{aligned}$$

and

$$\begin{aligned}
 R_{i,j}(q_{31}^*) &= \frac{\partial}{\partial \theta_{21}} I_{i,j}(q_{31}^*) \\
 &= -q_{31}^{*\frac{1}{2}f+i} p_{31}^{*f-1+j-r} / (\lambda + \theta_{21}) B(\tfrac{1}{2}f+i, f-1+j-r) . \quad (4.5.10)
 \end{aligned}$$

Using 4.5.7 through 4.5.10, we obtain, from 4.5.1, the partial derivative of $D_1(\lambda, \theta_{21}, \theta_{32})$ with respect to θ_{21} , i.e.,

$$\begin{aligned}
& \frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}, \theta_{32}) \\
&= - \frac{w_1(1-w_1)}{\theta_{21}^2} P(A_1'') \\
&+ \left[\frac{w_1(1-w_1)}{\theta_{21}} + w_2(1-w_2) + w_3(1-w_3)\theta_{32} \right] \frac{\partial}{\partial \theta_{21}} P(A_1'') \\
&+ G \left[\frac{w_1 w_2 I_{-\frac{1}{2}}(p_{21})}{2\theta_{21}} - w_1 w_2 R_{-\frac{1}{2}}(p_{21}) + \frac{w_1 w_3 \theta_{32}^{\frac{1}{2}}}{2\theta_{21}} \right] / \theta_{21}^{\frac{1}{2}} \\
&- \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} \frac{\partial}{\partial \theta_{21}} A(\lambda, \theta_{21}, \theta_{32}) \\
&- \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} \frac{\partial}{\partial \theta_{21}} B(\lambda, \theta_{21}, \theta_{32}) , \\
&\hspace{25em} (4.5.11)
\end{aligned}$$

where $I_i(p_{21})$ and $R_i(p_{21})$ are defined in 4.1.3 and 4.1.6 respectively,

$$\begin{aligned}
& \frac{\partial}{\partial \theta_{21}} P(A_1'') \\
&= - \sum_{r=0}^{\frac{1}{2}f-1} \binom{f-2-r}{\frac{1}{2}f-1-r} \left\{ q_{32}^{\frac{1}{2}f} p_{32}^{\frac{1}{2}f-1-r} R_{0,0}^c(p_{32}^*) - p_{21}^{\frac{1}{2}f} q_{21}^{\frac{1}{2}f-1-r} R_{0,0}(q_{31}^*) \right. \\
&\quad \left. - \frac{p_{21}^{\frac{1}{2}f} q_{21}^{\frac{1}{2}f-1-r} [\frac{1}{2}f q_{21} - (\frac{1}{2}f-1-r)p_{21}] I_{0,0}(q_{31}^*)}{\theta_{21}} \right\} , \\
&\hspace{25em} (4.5.12)
\end{aligned}$$

$$\frac{3}{2\theta_{21}}A(\lambda, \theta_{21}, \theta_{32})$$

$$\begin{aligned}
&= \left\{ w_1 w_2 p_{31}^{\frac{1}{2}f-\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f-\frac{1}{2} \right) q_{31} - \left(\frac{1}{2}f-1-r \right) p_{31}^{-\frac{1}{2}} \right] / \theta_{21}^{3/2} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f-\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f-\frac{1}{2} \right) p_{31} - \left(\frac{1}{2}f-1-r \right) q_{31} \right] / \theta_{21} \right\} \cdot \\
&\quad \left[I_{\frac{1}{2}, -\frac{1}{2}}(\bar{q}_{21}) - I_{\frac{1}{2}, -\frac{1}{2}}(q_{21}^*) \right] \\
&\quad + (w_1 w_2 p_{31}^{\frac{1}{2}f-\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f-\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r}) \cdot \\
&\quad \left[R_{\frac{1}{2}, -\frac{1}{2}}(\bar{q}_{21}) - R_{\frac{1}{2}, -\frac{1}{2}}(q_{21}^*) \right] \\
&\quad + \left\{ w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f-\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f-\frac{1}{2} \right) q_{21} - \left(\frac{1}{2}f-1-r \right) p_{21}^{-\frac{1}{2}} \right] / \theta_{21}^{3/2} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f-\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f-\frac{1}{2} \right) p_{21} - \left(\frac{1}{2}f-1-r \right) q_{21} \right] / \theta_{21} \right\} I_{\frac{1}{2}, -\frac{1}{2}}(q_{31}^*) \\
&\quad + (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f-\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f-\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r}) R_{\frac{1}{2}, -\frac{1}{2}}(q_{31}^*) \\
&\quad - (w_1 w_2 p_{32}^{\frac{1}{2}f-\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f-\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r}) I_{\frac{1}{2}, -\frac{1}{2}}^C(p_{32}^*) / 2\theta_{21}^{3/2} \\
&\quad + (w_1 w_2 p_{32}^{\frac{1}{2}f-\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f-\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r}) R_{\frac{1}{2}, -\frac{1}{2}}^C(p_{32}^*) / \theta_{21}^{\frac{1}{2}},
\end{aligned}$$

(4.5.15)

and

$$\begin{aligned}
& \frac{\partial}{\partial \theta_{21}} B(\lambda, \theta_{21}, \theta_{32}) \\
&= \left\{ w_1 w_2 p_{31}^{\frac{1}{2}f+\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) q_{31} - \left(\frac{1}{2}f-1-r \right) p_{31}^{-\frac{1}{2}} \right] / \theta_{21}^{3/2} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f+\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) p_{31} - \left(\frac{1}{2}f-1-r \right) q_{31} \right] / \theta_{21} \right\} \cdot \\
&\quad \left[I_{-\frac{1}{2}, \frac{1}{2}}(\bar{q}_{21}) - I_{-\frac{1}{2}, \frac{1}{2}}(q_{21}^*) \right] \\
&+ (w_1 w_2 p_{31}^{\frac{1}{2}f+\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f+\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r}) \cdot \\
&\quad \left[R_{-\frac{1}{2}, \frac{1}{2}}(\bar{q}_{21}) - R_{-\frac{1}{2}, \frac{1}{2}}(q_{21}^*) \right] \\
&+ \left\{ w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f+\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) q_{21} - \left(\frac{1}{2}f-1-r \right) p_{21}^{-\frac{1}{2}} \right] / \theta_{21}^{3/2} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f+\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) p_{21} - \left(\frac{1}{2}f-1-r \right) q_{21} \right] / \theta_{21} \right\} I_{-\frac{1}{2}, \frac{1}{2}}(q_{31}^*) \\
&+ (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f+\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f+\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r}) R_{-\frac{1}{2}, \frac{1}{2}}(q_{31}^*) \\
&- (w_1 w_2 p_{32}^{\frac{1}{2}f+\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f+\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r}) I_{-\frac{1}{2}, \frac{1}{2}}^c(p_{32}^*) / 2\theta_{21}^{3/2} \\
&+ (w_1 w_2 p_{32}^{\frac{1}{2}f+\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f+\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r}) R_{-\frac{1}{2}, \frac{1}{2}}^c(p_{32}^*) / \theta_{21}^{\frac{1}{2}}.
\end{aligned}
\tag{4.5.14}$$

It can be verified that

$$\begin{aligned}
\lim_{\theta_{21} \rightarrow \infty} \frac{\partial}{\partial \theta_{21}} P(A_1^n) &= \lim_{\theta_{21} \rightarrow \infty} \frac{\partial}{\partial \theta_{21}} A(\lambda, \theta_{21}, \theta_{32}) \\
&= \lim_{\theta_{21} \rightarrow \infty} \frac{\partial}{\partial \theta_{21}} B(\lambda, \theta_{21}, \theta_{32}) = 0.
\end{aligned}$$

Thus,

$$\lim_{\theta_{21} \rightarrow \infty} \frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}, \theta_{32}) = 0. \tag{4.5.15}$$

This implies that, as θ_{21} tends to infinity, the slope of

$D_1(\lambda, \theta_{21}, \theta_{32})$ is horizontal, i.e., $D_1(\lambda, \theta_{21}, \theta_{32})$ has a horizontal asymptote. Furthermore, it can be seen that, for a given $\lambda \geq 0$,

$$\lim_{\theta_{21} \rightarrow \infty} P(A_1'') = 1$$

and

$$\lim_{\theta_{21} \rightarrow \infty} A(\lambda, \theta_{21}, \theta_{32}) = \lim_{\theta_{21} \rightarrow \infty} B(\lambda, \theta_{21}, \theta_{32}) = 0.$$

Hence,

$$\lim_{\theta_{21} \rightarrow \infty} D_1(\lambda, \theta_{21}, \theta_{32}) = [w_2(1-w_2) + w_3(1-w_3)\theta_{32}] - Gw_2w_3\theta_{32}^{\frac{1}{2}}. \quad (4.5.16)$$

Therefore, for a given $\lambda \geq 0$, if θ_{32} is such that

$$w_2(1-w_2) + w_3(1-w_3)\theta_{32} - Gw_2w_3\theta_{32}^{\frac{1}{2}} > 0, \quad (4.5.17)$$

the limit of $D_1(\lambda, \theta_{21}, \theta_{32})$, as θ_{21} tends to infinity, is positive. This, in turn, implies that, for any given $\lambda \geq 0$ and θ_{32} satisfying 4.5.17, $D_1(\lambda, \theta_{21}, \theta_{32})$ has a positive horizontal asymptote. Thus, there exists a θ_{21}^0 such that $D_1(\lambda, \theta_{21}, \theta_{32}) > 0$ for each $\theta_{21} > \theta_{21}^0$. We, then, have proved the following theorem.

Theorem 4.9 For any given $\lambda \geq 0$, if θ_{32} satisfies 4.5.17, $\exists \theta_{21}^0$ such that

$$D_1(\lambda, \theta_{21}, \theta_{32}) > 0 \quad \forall \quad \theta_{21} > \theta_{21}^0.$$

Next, let us consider $D_1(\lambda, \theta_{21}, \theta_{32})/\theta_{32}$.

$$\begin{aligned}
& D_1(\lambda, \theta_{21}, \theta_{32}) / \theta_{32} \\
&= [w_1(1-w_1)/\theta_{21}\theta_{32} + w_2(1-w_2)/\theta_{32} + w_3(1-w_3)] P(A_1'') \\
&\quad - G[w_1 w_2 I_{-\frac{1}{2}}(p_{21}) / \theta_{21}^{\frac{1}{2}} \theta_{32} + w_1 w_3 / \theta_{21}^{\frac{1}{2}} \theta_{32}^{\frac{1}{2}} + w_2 w_3 / \theta_{32}^{\frac{1}{2}}] \\
&\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} A(\lambda, \theta_{21}, \theta_{32}) / \theta_{32} \\
&\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} B(\lambda, \theta_{21}, \theta_{32}) / \theta_{32}. \tag{4.5.18}
\end{aligned}$$

It can be seen that

$$\lim_{\theta_{32} \rightarrow \infty} P(A_1'') = 1$$

and

$$\lim_{\theta_{32} \rightarrow \infty} \frac{A(\lambda, \theta_{21}, \theta_{32})}{\theta_{32}} = \lim_{\theta_{32} \rightarrow \infty} \frac{B(\lambda, \theta_{21}, \theta_{32})}{\theta_{32}} = 0.$$

Then,

$$\lim_{\theta_{32} \rightarrow \infty} \frac{D_1(\lambda, \theta_{21}, \theta_{32})}{\theta_{32}} = w_3(1-w_3) > 0. \tag{4.5.19}$$

Therefore, as θ_{32} tends to infinity, $D_1(\lambda, \theta_{21}, \theta_{32})$ tends to a positive limit. Hence, there exists a θ_{32}^0 such that, for any given $\lambda \geq 0$ and any $\theta_{21} \geq 1$, $D_1(\lambda, \theta_{21}, \theta_{32}) > 0$ for each $\theta_{32} > \theta_{32}^0$. The following theorem is proved.

Theorem 4.10 For any given $\lambda \geq 0$ and $\theta_{21} \geq 1$,
 $\exists \theta_{32}^0$ such that

$$D_1(\lambda, \theta_{21}, \theta_{32}) > 0 \quad \forall \theta_{32} > \theta_{32}^0.$$

We shall now discuss the behavior of $D_1(\lambda, \theta_{21}, \theta_{32})$ as λ varies. Consider first the limit of $D_1(\lambda, \theta_{21}, \theta_{32})$, as λ tends to zero, i.e.,

$$\begin{aligned}
\lim_{\lambda \rightarrow 0} D_1(\lambda, \theta_{21}, \theta_{32}) &= D_1(0, \theta_{21}, \theta_{32}) \\
&= \frac{w_1 w_2}{\theta_{21}} (\theta_{21} - G \theta_{21}^{\frac{1}{2}} + 1) + w_2 w_3 (\theta_{32} - G \theta_{32}^{\frac{1}{2}} + 1) \\
&\quad + \frac{w_1 w_3}{\theta_{21}} (\theta_{21} \theta_{32} - G \theta_{21}^{\frac{1}{2}} \theta_{32}^{\frac{1}{2}} + 1). \quad (4.5.20)
\end{aligned}$$

If both θ_{21} and θ_{32} are larger than θ_m where θ_m is defined in 4.1.10, $D_1(\lambda, \theta_{21}, \theta_{32})$ will have a positive limit, as λ tends to zero. Let $Q_{i,j}(x)$ and $Q_{i,j}^c(x)$ denote partial derivatives of $I_{i,j}(x)$ and $I_{i,j}^c(x)$ with respect to λ respectively. Then

$$\begin{aligned}
Q_{i,j}(\bar{q}_{21}) &= \frac{\partial}{\partial \lambda} I_{i,j}(\bar{q}_{21}) \\
&= \frac{\partial}{\partial \lambda} I_{\bar{q}_{21}}(\tfrac{1}{2}f+i, f-1+j-r) \\
&= \frac{\partial}{\partial \lambda} \int_0^{\bar{q}_{21}} x^{\frac{1}{2}f+i-1} (1-x)^{f-2+j-r} dx / B(\tfrac{1}{2}f+i, f-1+j-r) \\
&= \frac{\frac{1}{2}f+i-1}{\bar{q}_{21}} \bar{q}_{21}^{f-2+j-r} / \lambda^2 B(\tfrac{1}{2}f+i, f-1+j-r). \quad (4.5.21)
\end{aligned}$$

Similarly,

$$\begin{aligned}
Q_{i,j}(q_{21}^*) &= \frac{\partial}{\partial \lambda} I_{i,j}(q_{21}^*) \\
&= -q_{21}^{\frac{1}{2}f+i+1} p_{21}^{f-2+j-r} / \theta_{32} B(\tfrac{1}{2}f+i, f-1+j-r), \quad (4.5.22)
\end{aligned}$$

$$\begin{aligned}
Q_{i,j}^c(p_{32}^*) &= \frac{\partial}{\partial \lambda} I_{i,j}^c(p_{32}^*) \\
&= p_{32}^{f-1+j-r} q_{32}^{\frac{1}{2}f+i} / (\lambda + \theta_{32}) B(f-1+j-r, \tfrac{1}{2}f+i), \quad (4.5.23)
\end{aligned}$$

and

$$\begin{aligned}
Q_{i,j}(q_{31}^*) &= \frac{\partial}{\partial \lambda} I_{i,j}(q_{31}^*) \\
&= q_{31}^{*\frac{1}{2}f+i} p_{31}^{*f-1+j-r} / \lambda (1+\lambda/\theta_{21}) B(\frac{1}{2}f+i, f-1+j-r) .
\end{aligned}
\tag{4.5.24}$$

Using 4.5.1 and 4.5.21 through 4.5.24, we obtain the partial derivative of $D_1(\lambda, \theta_{21}, \theta_{32})$ with respect to λ , i.e.,

$$\begin{aligned}
\frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}, \theta_{32}) &= [w_1(1-w_1)/\theta_{21} + w_2(1-w_2) + w_3(1-w_3)\theta_{32}] \frac{\partial}{\partial \lambda} P(A_1'') \\
&\quad - G w_1 w_2 Q_{-\frac{1}{2}}(p_{21}) / \theta_{21}^{\frac{1}{2}} \\
&\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} \frac{\partial}{\partial \lambda} A(\lambda, \theta_{21}, \theta_{32}) \\
&\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} \frac{\partial}{\partial \lambda} B(\lambda, \theta_{21}, \theta_{32}) ,
\end{aligned}
\tag{4.5.25}$$

where

$$\begin{aligned}
&\frac{\partial}{\partial \lambda} P(A_1'') \\
&= - \sum_{r=0}^{\frac{1}{2}f-1} \binom{f-2-r}{\frac{1}{2}f-1-r} \left\{ q_{32}^{\frac{1}{2}f} p_{32}^{\frac{1}{2}f-1-r} [\frac{1}{2}f p_{32} - (\frac{1}{2}f-1-r) q_{32}] I_{0,0}^c(p_{32}^*) / \lambda \right. \\
&\quad + q_{32}^{\frac{1}{2}f} p_{32}^{\frac{1}{2}f-1-r} Q_{0,0}^c(p_{32}^*) \\
&\quad + p_{21}^{\frac{1}{2}f} q_{21}^{\frac{1}{2}f-1-r} [\frac{1}{2}f q_{21} - (\frac{1}{2}f-1-r) p_{21}] I_{0,0}(q_{31}^*) / \lambda \\
&\quad \left. - p_{21}^{\frac{1}{2}f} q_{21}^{\frac{1}{2}f-1-r} Q_{0,0}(q_{31}^*) \right\} ,
\end{aligned}
\tag{4.5.26}$$

$$\begin{aligned}
& \frac{\partial}{\partial \lambda} A(\lambda, \theta_{21}, \theta_{32}) \\
&= - \left\{ w_1 w_2 p_{31}^{\frac{1}{2}f - \frac{1}{2}} q_{31}^{\frac{1}{2}f - 1 - r} \left[\left(\frac{1}{2}f - \frac{1}{2} \right) q_{31} - \left(\frac{1}{2}f - 1 - r \right) p_{31} \right] / \lambda \theta_{21}^{\frac{1}{2}} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f - \frac{1}{2}} p_{31}^{\frac{1}{2}f - 1 - r} \left[\left(\frac{1}{2}f - \frac{1}{2} \right) p_{31} - \left(\frac{1}{2}f - 1 - r \right) q_{31} \right] / \lambda \right\} \\
&\quad [I_{\frac{1}{2}, -\frac{1}{2}}(\bar{q}_{21}) - I_{\frac{1}{2}, -\frac{1}{2}}(q_{21}^*)] \\
&\quad + (w_1 w_2 p_{31}^{\frac{1}{2}f - \frac{1}{2}} q_{31}^{\frac{1}{2}f - 1 - r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f - \frac{1}{2}} p_{31}^{\frac{1}{2}f - 1 - r}) \cdot \\
&\quad [Q_{\frac{1}{2}, -\frac{1}{2}}(\bar{q}_{21}) - Q_{\frac{1}{2}, -\frac{1}{2}}(q_{21}^*)] \\
&\quad - \left\{ w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f - \frac{1}{2}} q_{21}^{\frac{1}{2}f - 1 - r} \left[\left(\frac{1}{2}f - \frac{1}{2} \right) q_{21} - \left(\frac{1}{2}f - 1 - r \right) p_{21} \right] / \lambda \theta_{21}^{\frac{1}{2}} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f - \frac{1}{2}} p_{21}^{\frac{1}{2}f - 1 - r} \left[\left(\frac{1}{2}f - \frac{1}{2} \right) p_{21} - \left(\frac{1}{2}f - 1 - r \right) q_{21} \right] / \lambda \right\} I_{\frac{1}{2}, -\frac{1}{2}}(q_{31}^*) \\
&\quad + (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f - \frac{1}{2}} q_{21}^{\frac{1}{2}f - 1 - r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f - \frac{1}{2}} p_{21}^{\frac{1}{2}f - 1 - r}) Q_{\frac{1}{2}, -\frac{1}{2}}(q_{31}^*) \\
&\quad - \left\{ w_1 w_2 p_{32}^{\frac{1}{2}f - \frac{1}{2}} q_{32}^{\frac{1}{2}f - 1 - r} \left[\left(\frac{1}{2}f - \frac{1}{2} \right) q_{32} - \left(\frac{1}{2}f - 1 - r \right) p_{32} \right] \right. \\
&\quad \left. + w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f - \frac{1}{2}} p_{32}^{\frac{1}{2}f - 1 - r} \left[\left(\frac{1}{2}f - \frac{1}{2} \right) p_{32} - \left(\frac{1}{2}f - 1 - r \right) q_{32} \right] \right\} \frac{I_{\frac{1}{2}, -\frac{1}{2}}^c(p_{32}^*)}{\lambda \theta_{21}^{\frac{1}{2}}} \\
&\quad + (w_1 w_2 p_{32}^{\frac{1}{2}f - \frac{1}{2}} q_{32}^{\frac{1}{2}f - 1 - r} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f - \frac{1}{2}} p_{32}^{\frac{1}{2}f - 1 - r}) Q_{\frac{1}{2}, -\frac{1}{2}}^c(p_{32}^*) / \theta_{21}^{\frac{1}{2}} , \\
&\hspace{25em} (4.5.27)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial \lambda} B(\lambda, \theta_{21}, \theta_{32}) \\
&= - \left\{ w_1 w_2 p_{31}^{\frac{1}{2}f+\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) q_{31} - \left(\frac{1}{2}f-1-r \right) p_{31} \right] / \lambda \theta_{21}^{\frac{1}{2}} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f+\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) p_{31} - \left(\frac{1}{2}f-1-r \right) q_{31} \right] / \lambda \right\} \cdot \\
&\quad \left[I_{-\frac{1}{2}, \frac{1}{2}}(\bar{q}_{21}) - I_{-\frac{1}{2}, \frac{1}{2}}(q_{21}^*) \right] \\
&+ (w_1 w_2 p_{31}^{\frac{1}{2}f+\frac{1}{2}} q_{31}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{31}^{\frac{1}{2}f+\frac{1}{2}} p_{31}^{\frac{1}{2}f-1-r}) \cdot \\
&\quad \left[Q_{-\frac{1}{2}, \frac{1}{2}}(\bar{q}_{21}) - Q_{-\frac{1}{2}, \frac{1}{2}}(q_{21}^*) \right] \\
&- \left\{ w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f+\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) q_{21} - \left(\frac{1}{2}f-1-r \right) p_{21} \right] / \lambda \theta_{21}^{\frac{1}{2}} \right. \\
&\quad \left. + w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f+\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) p_{21} - \left(\frac{1}{2}f-1-r \right) q_{21} \right] / \lambda \right\} I_{-\frac{1}{2}, \frac{1}{2}}(q_{31}^*) \\
&+ (w_1 w_3 \theta_{32}^{\frac{1}{2}} p_{21}^{\frac{1}{2}f+\frac{1}{2}} q_{21}^{\frac{1}{2}f-1-r} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} q_{21}^{\frac{1}{2}f+\frac{1}{2}} p_{21}^{\frac{1}{2}f-1-r}) Q_{-\frac{1}{2}, \frac{1}{2}}(q_{31}^*) \\
&- \left\{ w_1 w_2 p_{32}^{\frac{1}{2}f+\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) q_{32} - \left(\frac{1}{2}f-1-r \right) p_{32} \right] \right. \\
&\quad \left. + w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f+\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r} \left[\left(\frac{1}{2}f+\frac{1}{2} \right) p_{32} - \left(\frac{1}{2}f-1-r \right) q_{32} \right] \right\} \frac{I_{-\frac{1}{2}, \frac{1}{2}}^c(p_{32}^*)}{\lambda \theta_{21}^{\frac{1}{2}}} \\
&+ (w_1 w_2 p_{32}^{\frac{1}{2}f+\frac{1}{2}} q_{32}^{\frac{1}{2}f-1-r} - w_1 w_3 \theta_{32}^{\frac{1}{2}} q_{32}^{\frac{1}{2}f+\frac{1}{2}} p_{32}^{\frac{1}{2}f-1-r}) Q_{-\frac{1}{2}, \frac{1}{2}}^c(p_{32}^*) / \theta_{21}^{\frac{1}{2}}.
\end{aligned}$$

(4.5.28)

It can be verified that

$$\begin{aligned}
& \lim_{\lambda \rightarrow \infty} [w_1(1-w_1)/\theta_{21} + w_2(1-w_2) + w_3(1-w_3)\theta_{32}] \frac{\partial}{\partial \lambda} P(A_1'') \\
&= c(\theta_{21}, \theta_{32}) 0^+,
\end{aligned}$$

$$\lim_{\lambda \rightarrow \infty} w_1 w_2 Q_{-\frac{1}{2}}(p_{21}) / \theta_{21}^{\frac{1}{2}} = d(\theta_{21}) 0^+,$$

$$\lim_{\lambda \rightarrow \infty} \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} \frac{\partial}{\partial \lambda} A(\lambda, \theta_{21}, \theta_{32}) = a(\theta_{21}, \theta_{32}) 0^+,$$

and

$$\lim_{\lambda \rightarrow \infty} \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} \frac{\partial}{\partial \lambda} B(\lambda, \theta_{21}, \theta_{32}) = b(\theta_{21}, \theta_{32}) 0^+,$$

where

$$c(\theta_{21}, \theta_{32})$$

$$= [w_1(1-w_1)/\theta_{21} + w_2(1-w_2) + w_3(1-w_3)\theta_{32}].$$

$$\begin{aligned} \sum_{r=0}^{\frac{1}{2}f-1} \binom{f-2-r}{\frac{1}{2}f-1-r} & \left\{ \frac{1}{2}f-1-r-\frac{1}{2}f I_{0,0}\left(\frac{1}{1+\theta_{32}}\right) \right. \\ & \left. - [1-\theta_{32}^{f-1-r}/(1+\theta_{32})^{3f/2-1-r}]/B(\frac{1}{2}f, f-1-r) \right\}, \end{aligned} \quad (4.5.29)$$

$$d(\theta_{21}) = -w_1 w_2 / \theta_{21}^{\frac{1}{2}} B(\frac{1}{2}f-\frac{1}{2}, \frac{1}{2}f-\frac{1}{2}), \quad (4.5.30)$$

$$a(\theta_{21}, \theta_{32}) = \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} [a_1(\theta_{21}, \theta_{32}) + a_2(\theta_{21}, \theta_{32})], \quad (4.5.31)$$

with

$$\begin{aligned} a_1(\theta_{21}, \theta_{32}) &= -[w_1 w_2 (\frac{1}{2}f-\frac{1}{2}) / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f-1-r)] I_{\frac{1}{2}, -\frac{1}{2}}\left(\frac{\theta_{32}}{1+\theta_{32}}\right) \\ &\quad - [w_1 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f-\frac{1}{2}) / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f-1-r)] I_{\frac{1}{2}, -\frac{1}{2}}\left(\frac{1}{1+\theta_{32}}\right) \\ &\quad - [w_1 w_2 (\frac{1}{2}f-\frac{1}{2}) - w_1 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f-1-r)] / \theta_{21}^{\frac{1}{2}}, \end{aligned} \quad (4.5.32)$$

and

$$\begin{aligned}
& a_2(\theta_{21}, \theta_{32}) \\
&= \left\{ (w_1 w_2 / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}}) \left[\left(\frac{\theta_{32}}{1+\theta_{32}} \right)^{\frac{1}{2}f+3/2} \left(\frac{1}{1+\theta_{32}} \right)^{f-5/2-r} \theta_{21+1/\theta_{32}} \right] \right. \\
&\quad + (w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}}) \left(\frac{1}{1+\theta_{32}} \right)^{\frac{1}{2}f+\frac{1}{2}} \left(\frac{\theta_{32}}{1+\theta_{32}} \right)^{f-3/2-r} \\
&\quad \left. + (w_1 w_2 - w_1 w_3 \theta_{32}^{\frac{1}{2}}) / \theta_{21}^{\frac{1}{2}} \right\} / B(\frac{1}{2}f+\frac{1}{2}, f-3/2-r), \quad (4.5.33)
\end{aligned}$$

and

$$b(\theta_{21}, \theta_{32}) = \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} [b_1(\theta_{21}, \theta_{32}) + b_2(\theta_{21}, \theta_{32})], \quad (4.5.34)$$

with

$$\begin{aligned}
b_1(\theta_{21}, \theta_{32}) &= -[w_1 w_2 (\frac{1}{2}f+\frac{1}{2}) / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f-1-r)] I_{-\frac{1}{2}, \frac{1}{2}} \left(\frac{\theta_{32}}{1+\theta_{32}} \right) \\
&\quad - [w_1 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f+\frac{1}{2}) / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f-1-r)] I_{-\frac{1}{2}, \frac{1}{2}} \left(\frac{1}{1+\theta_{32}} \right) \\
&\quad - [w_1 w_2 (\frac{1}{2}f+\frac{1}{2}) - w_2 w_3 \theta_{32}^{\frac{1}{2}} (\frac{1}{2}f-1-r)] / \theta_{21}^{\frac{1}{2}}, \quad (4.5.35)
\end{aligned}$$

and

$$\begin{aligned}
& b_2(\theta_{21}, \theta_{32}) \\
&= \left\{ (w_1 w_2 / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}}) \left[\left(\frac{\theta_{32}}{1+\theta_{32}} \right)^{\frac{1}{2}f+\frac{1}{2}} \left(\frac{1}{1+\theta_{32}} \right)^{f-3/2-r} \theta_{21+1/\theta_{32}} \right] \right. \\
&\quad + (w_1 w_3 \theta_{32}^{\frac{1}{2}} / \theta_{21}^{\frac{1}{2}} - w_2 w_3 \theta_{32}^{\frac{1}{2}}) \left(\frac{1}{1+\theta_{32}} \right)^{\frac{1}{2}f-\frac{1}{2}} \left(\frac{\theta_{32}}{1+\theta_{32}} \right)^{f-\frac{1}{2}-r} \\
&\quad \left. + (w_1 w_2 - w_1 w_3 \theta_{32}^{\frac{1}{2}}) / \theta_{21}^{\frac{1}{2}} \right\} / B(\frac{1}{2}f-\frac{1}{2}, f-\frac{1}{2}-r). \quad (4.5.36)
\end{aligned}$$

Using 4.5.29 through 4.5.36, we have

$$\begin{aligned}
& \lim_{\lambda \rightarrow \infty} \frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}, \theta_{32}) \\
&= [c(\theta_{21}, \theta_{32}) - Gd(\theta_{21}) - \frac{1}{2}Ga(\theta_{21}, \theta_{32}) - \frac{1}{2}Gb(\theta_{21}, \theta_{32})] 0^+.
\end{aligned}$$

If θ_{21} and θ_{32} are so chosen that

$$c(\theta_{21}, \theta_{32}) - Gd(\theta_{21}) - \frac{1}{2}Ga(\theta_{21}, \theta_{32}) - \frac{1}{2}Gb(\theta_{21}, \theta_{32}) > 0, \quad (4.5.37)$$

then $\lim_{\lambda \rightarrow \infty} \frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}, \theta_{32}) = 0^+$. This means that given θ_{21} and θ_{32} satisfying 4.5.37, $D_1(\lambda, \theta_{21}, \theta_{32})$ tends to its horizontal asymptote from below, as λ tends to infinity. Furthermore, since

$$\lim_{\lambda \rightarrow \infty} D_1(\lambda, \theta_{21}, \theta_{32}) = 0, \quad (4.5.38)$$

$D_1(\lambda, \theta_{21}, \theta_{32})$ tends to zero from below, as λ tends to infinity, i.e., because $\lim_{\lambda \rightarrow \infty} \frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}, \theta_{32}) = 0^+$. Hence, there exists a λ^0 such that $D_1(\lambda, \theta_{21}, \theta_{32}) > 0$ for each $\lambda < \lambda^0$. Then the following theorem is proved.

Theorem 4.11 For any given $\theta_{21} > \theta_m$ and $\theta_{32} > \theta_m$ satisfying 4.5.37, $\exists \lambda^0$ such that

$$D_1(\lambda, \theta_{21}, \theta_{32}) > 0 \quad \forall \lambda < \lambda^0.$$

F. Comparison of Sometimes Proportional

Allocation with Modified Neyman

Allocation for $k=3$ and $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2$

Consider the difference function

$$\begin{aligned} D_2(\lambda, \theta_{21}, \theta_{32}) &= [V(\bar{y}_w)_N - V(\bar{y}_w)_S] / \frac{\sigma_2^2}{n} \\ &= G w_1 w_2 I_{-\frac{1}{2}}(q_{21}) / \theta_{21}^{\frac{1}{2}} \\ &\quad - [w_1(1-w_1)/\theta_{21} + w_2(1-w_2) + w_3(1-w_3)\theta_{32}] P(A_0'') \\ &\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-3/2-r)}{\Gamma(\frac{1}{2}f-\frac{1}{2})\Gamma(\frac{1}{2}f-r)} A(\lambda, \theta_{21}, \theta_{32}) \\ &\quad - \frac{1}{2} G \sum_{r=0}^{\frac{1}{2}f-1} \frac{\Gamma(f-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}f+\frac{1}{2})\Gamma(\frac{1}{2}f-r)} B(\lambda, \theta_{21}, \theta_{32}), \end{aligned} \quad (4.6.1)$$

where $P(A_0^n)$, $A(\lambda, \theta_{21}, \theta_{32})$ and $B(\lambda, \theta_{21}, \theta_{32})$ are defined in 4.5.4, 4.5.5 and 4.5.6 respectively. $D_2(\lambda, \theta_{21}, \theta_{32})$ can also be expressed in terms of $D_1(\lambda, \theta_{21}, \theta_{32})$ as follow.

$$\begin{aligned} D_2(\lambda, \theta_{21}, \theta_{32}) &= \left\{ V(\bar{y}_w)_P - V(\bar{y}_w)_S - [V(\bar{y}_w)_P - V(\bar{y}_w)_N] \right\} / \frac{\sigma_2^2}{n} \\ &= D_1(\lambda, \theta_{21}, \theta_{32}) - D_1(0, \theta_{21}, \theta_{32}), \end{aligned} \quad (4.6.2)$$

where $D_1(\lambda, \theta_{21}, \theta_{32})$ is defined in 4.5.1 and $D_1(0, \theta_{21}, \theta_{32})$ is defined in 4.5.19.

Clearly, if $D_2(\lambda, \theta_{21}, \theta_{32}) > 0$ sometimes proportional allocation is more efficient than modified Neyman allocation. Otherwise, modified Neyman allocation is more efficient.

Let us consider the case that λ is a fixed but arbitrary non-negative number. We shall discuss the behavior of $D_2(\lambda, \theta_{21}, \theta_{32})$ as θ_{21} and θ_{32} vary. From 4.6.2, we obtain the partial derivative of $D_2(\lambda, \theta_{21}, \theta_{32})$ with respect to θ_{21} , i.e.,

$$\frac{\partial}{\partial \theta_{21}} D_2(\lambda, \theta_{21}, \theta_{32}) = \frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}, \theta_{32}) - \frac{\partial}{\partial \theta_{21}} D_1(0, \theta_{21}, \theta_{32}), \quad (4.6.3)$$

where $\frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}, \theta_{32})$ is defined in 4.5.11

and

$$\frac{\partial}{\partial \theta_{21}} D_1(0, \theta_{21}, \theta_{32}) = - \frac{w_1(1-w_1)}{\theta_{21}^2} + \frac{1}{2} G(w_1 w_2 + w_1 w_3 \theta_{32}^{\frac{1}{2}}) / \theta_{21}^{5/2}. \quad (4.6.4)$$

From 4.5.15, we can see that

$$\lim_{\theta_{21} \rightarrow \infty} \frac{\partial}{\partial \theta_{21}} D_2(\lambda, \theta_{21}, \theta_{32}) = 0.$$

Therefore, as θ_{21} tends to infinity, $D_2(\lambda, \theta_{21}, \theta_{32})$ has a horizontal asymptote. And from 4.5.16, we have

$$\begin{aligned}
& \lim_{\theta_{21} \rightarrow \infty} D_2(\lambda, \theta_{21}, \theta_{32}) \\
&= \lim_{\theta_{21} \rightarrow \infty} D_1(\lambda, \theta_{21}, \theta_{32}) - \lim_{\theta_{21} \rightarrow \infty} D_1(0, \theta_{21}, \theta_{32}) \\
&= w_2(1-w_2) + w_3(1-w_3)\theta_{32} - Gw_2w_3\theta_{32}^{\frac{1}{2}} \\
&\quad - \lim_{\theta_{21} \rightarrow \infty} [w_1(1-w_1)/\theta_{21} + w_2(1-w_2) + w_3(1-w_3)\theta_{32} \\
&\quad \quad - G(w_1w_2/\theta_{21}^{\frac{1}{2}} + w_1w_3\theta_{32}^{\frac{1}{2}}/\theta_{21}^{\frac{1}{2}} + w_2w_3\theta_{32}^{\frac{1}{2}})] \\
&= w_1[G(w_2 + w_3\theta_{32}^{\frac{1}{2}}) - (1-w_1)] 0^+ \\
&= 0^+.
\end{aligned}$$

Hence, $D_2(\lambda, \theta_{21}, \theta_{32})$ has the horizontal axis as its horizontal asymptote and tends to its asymptote from above.

There exists a θ_{21}^0 such that $D_2(\lambda, \theta_{21}, \theta_{32}) > 0$ for each $\theta_{21} < \theta_{21}^0$. We have now proved the following theorem.

Theorem 4.12 For any given $\lambda \geq 0$ and $\theta_{32} \geq 1$,
 $\exists \theta_{21}^0$ such that

$$D_2(\lambda, \theta_{21}, \theta_{32}) > 0 \quad \forall \quad \theta_{21} < \theta_{21}^0.$$

Next let us consider $D_2(\lambda, \theta_{21}, \theta_{32})/\theta_{32}$, i.e.,

$$D_2(\lambda, \theta_{21}, \theta_{32})/\theta_{32} = D_1(\lambda, \theta_{21}, \theta_{32})/\theta_{32} - D_1(0, \theta_{21}, \theta_{32})/\theta_{32}, \quad (4.6.5)$$

where $D_1(\lambda, \theta_{21}, \theta_{32})/\theta_{32}$ is given in 4.5.18

and

$$\begin{aligned}
D_1(0, \theta_{21}, \theta_{32})/\theta_{32} &= w_1(1-w_1)/\theta_{21}\theta_{32} + w_2(1-w_2)/\theta_{32} + w_3(1-w_3) \\
&\quad - G(w_1w_2/\theta_{21}^{\frac{1}{2}}\theta_{32} + w_1w_3/\theta_{21}^{\frac{1}{2}}\theta_{32}^{\frac{1}{2}} + w_2w_3/\theta_{32}^{\frac{1}{2}}).
\end{aligned} \quad (4.6.6)$$

From 4.5.19, 4.6.5 and 4.6.6, we obtain

$$\lim_{\theta_{32} \rightarrow \infty} \frac{D_2(\lambda, \theta_{21}, \theta_{32})}{\theta_{32}} = w_3(1-w_3) \left[\lim_{\theta_{32} \rightarrow \infty} P(A_1^n) - 1 \right] = 0^-. \quad (4.6.7)$$

4.6.7 implies that $D_2(\lambda, \theta_{21}, \theta_{32})$ tends to a negative limit, as θ_{32} tends to infinity. Therefore, there exists a θ_{32}^0 such that $D_2(\lambda, \theta_{21}, \theta_{32}) > 0$ for each $\theta_{32} < \theta_{32}^0$. We thus proved the following theorem.

Theorem 4.13 For any given $\lambda \geq 0$ and $\theta_{21} \geq 1$,
 $\exists \theta_{32}^0$ such that

$$D_2(\lambda, \theta_{21}, \theta_{32}) > 0 \quad \forall \theta_{32} < \theta_{32}^0.$$

We shall now discuss the behavior of $D_2(\lambda, \theta_{21}, \theta_{32})$ as λ varies. From 4.5.20 and 4.5.38, we can see that if both θ_{21} and θ_{32} are less than θ_m where θ_m is defined in 4.1.10, then

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} D_2(\lambda, \theta_{21}, \theta_{32}) &= \lim_{\lambda \rightarrow \infty} D_1(\lambda, \theta_{21}, \theta_{32}) - D_1(0, \theta_{21}, \theta_{32}) \\ &= -D_1(0, \theta_{21}, \theta_{32}) > 0. \end{aligned}$$

Since $\frac{\partial}{\partial \lambda} D_2(\lambda, \theta_{21}, \theta_{32}) = \frac{\partial}{\partial \lambda} D_1(\lambda, \theta_{21}, \theta_{32})$,

thus, if

$$c(\theta_{21}, \theta_{32}) - Gd(\theta_{21}) - \frac{1}{2}Ga(\theta_{21}, \theta_{32}) - \frac{1}{2}Gb(\theta_{21}, \theta_{32}) < 0, \quad (4.6.8)$$

then $\lim_{\lambda \rightarrow \infty} \frac{\partial}{\partial \lambda} D_2(\lambda, \theta_{21}, \theta_{32}) = 0^-$. Therefore, $D_2(\lambda, \theta_{21}, \theta_{32})$ tends to its positive horizontal asymptote from above, as λ tends to infinity. Thus, we have the following theorem.

Theorem 4.14 For given $\theta_{21} < \theta_m$ and $\theta_{32} < \theta_m$ satisfying 4.6.8, $\exists \lambda^0$ such that

$$D_2(\lambda, \theta_{21}, \theta_{32}) > 0 \quad \forall \lambda > \lambda^0.$$

V. EFFICIENCY OF SOMETIMES PROPORTIONAL ALLOCATION

The relative efficiency of sometimes proportional allocation with respect to proportional allocation and modified Neyman allocation will be discussed in this section. When $k > 2$, the expression for the variance of the estimate \bar{y}_w under sometimes proportional allocation becomes very complicated. Therefore, for relative efficiency of sometimes proportional allocation, we shall consider the case for $k=2$ only.

A. Efficiency with Respect to Proportional Allocation for $\sigma_1^2 \leq \sigma_2^2$

Consider the relative efficiency of sometimes proportional allocation with respect to proportional allocation,

$$\begin{aligned} e_1(\lambda, \theta_{21}) &= V(\bar{y}_w)_P / V(\bar{y}_w)_S \\ &= 1 / \left[1 - \frac{w_1 w_2}{w_1 + w_2 \theta_{21}} D_1(\lambda, \theta_{21}) \right], \end{aligned} \quad (5.1.1)$$

where $D_1(\lambda, \theta_{21})$ is defined in 4.1.1.

Clearly, if $e_1(\lambda, \theta_{21}) > 1$, sometimes proportional allocation is more efficient than proportional allocation. From 5.1.1, we see that the behavior of $e_1(\lambda, \theta_{21})$ is closely related to that of $D_1(\lambda, \theta_{21})$. In particular, when θ_{21} is fixed but arbitrary, the behavior of $e_1(\lambda, \theta_{21})$ will be the same as that of $D_1(\lambda, \theta_{21})$.

Since $e_1(\lambda, \theta_{21})$ and $D_1(\lambda, \theta_{21})$ are closely related,

some of the results of $e_1(\lambda, \theta_{21})$ parallel to those of $D_1(\lambda, \theta_{21})$ which have been presented in section IV A can be obtained quite easily. If a result of $e_1(\lambda, \theta_{21})$ is merely an immediate consequence of its analogue of $D_1(\lambda, \theta_{21})$ by simply observing the relationship between $e_1(\lambda, \theta_{21})$ and $D_1(\lambda, \theta_{21})$ given in 5.1.1, then we shall state the result without proof. First, let us establish a lemma parallel to Lemma 4.1.

Lemma 5.1 For any $\lambda \geq 0$, $\lim_{\theta_{21} \rightarrow 1} e_1(\lambda, \theta_{21}) \leq 1$.

Furthermore, $e_1(\lambda, 1)$ is an increasing function in λ except when $\lambda=1$.

Proof From 5.1.1

$$\lim_{\theta_{21} \rightarrow 1} e_1(\lambda, \theta_{21}) = e_1(\lambda, 1) = 1/[1 - w_1 w_2 D_1(\lambda, 1)].$$

By Lemma 4.1, we get

$$\lim_{\theta_{21} \rightarrow 1} e_1(\lambda, \theta_{21}) \leq 1.$$

As we mentioned earlier, $e_1(\lambda, 1)$ and $D_1(\lambda, 1)$ will behave in the same manner. We have seen, in Lemma 4.1, that

$$\frac{\partial}{\partial \lambda} D_1(\lambda, 1) = \lambda^{\frac{1}{2}f-3/2} (1-\lambda^{\frac{1}{2}})^2 / (1+\lambda)^f B(\frac{1}{2}f, \frac{1}{2}f)$$

which is positive except when $\lambda=1$. Thus $e_1(\lambda, 1)$ is an increasing function in λ except when $\lambda=1$. This completes the proof of the lemma.

Next let us consider the case that λ is a fixed but arbitrary non-negative number. We shall discuss the behavior

of $e_1(\lambda, \theta_{21})$ as θ_{21} varies by first proving a lemma analogous to Lemma 4.2.

Lemma 5.2 For any given $\lambda \geq 0$, satisfying 4.1.7, $\exists \theta'$ such that

$$\frac{\partial}{\partial \theta_{21}} e_1(\lambda, \theta_{21}) > 0 \quad \text{for each } \theta_{21} > \theta'.$$

Proof Taking partial derivative of $e_1(\lambda, \theta_{21})$, given in 5.1.1, with respect to θ_{21} , we obtain

$$\begin{aligned} \frac{\partial}{\partial \theta_{21}} e_1(\lambda, \theta_{21}) &= \frac{w_1 w_2}{w_1 + w_2 \theta_{21}} \left[\frac{\partial}{\partial \theta_{21}} D_1(\lambda, \theta_{21}) \right. \\ &\quad \left. - \frac{w_2}{w_1 + w_2 \theta_{21}} D_1(\lambda, \theta_{21}) \right] / \left[1 - \frac{w_1 w_2}{w_1 + w_2 \theta_{21}} D_1(\lambda, \theta_{21}) \right]^2. \end{aligned} \quad (5.1.2)$$

Substituting from 4.1.1 and 4.1.8 in 5.1.2 and taking limit of $\frac{1}{\theta_{21}} \frac{\partial}{\partial \theta_{21}} e_1(\lambda, \theta_{21})$, as θ_{21} tends to infinity, we have

$$\lim_{\theta_{21} \rightarrow \infty} \frac{1}{\theta_{21}} \frac{\partial}{\partial \theta_{21}} e_1(\lambda, \theta_{21}) = w_1 \left[1 - \frac{1}{2}G + \frac{\lambda^{\frac{1}{2}f - \frac{1}{2}} (\lambda^{\frac{1}{2}} - 1)}{B(\frac{1}{2}f, \frac{1}{2}f)} - 2 + G \right] 0^+ = 0^+.$$

Therefore, there must exists θ' such that

$$\frac{\partial}{\partial \theta_{21}} e_1(\lambda, \theta_{21}) > 0 \quad \text{for each } \theta_{21} > \theta'.$$

This completes the proof of the lemma.

We are now ready to establish the following theorem analogous to Theorem 4.1.

Theorem 5.1 For any $\lambda \geq 0$ satisfying 4.1.7,

$\exists \theta_0$ such that

$$e_1(\lambda, \theta_0) = 1$$

and

$$e_1(\lambda, \theta_{21}) > 1 \quad \text{for each } \theta_{21} > \theta_0.$$

Proof This is a direct consequence of Lemmas

5.1 and 5.2.

For any given $\lambda \geq 0$ satisfying 4.1.7, Theorem 5.1 assures us there exists a θ_0 such that for $\theta_{21} \geq \theta_0$ sometimes proportional allocation is always more efficient than proportional allocation.

Taking, in particular, $\lambda=0$ we obtain the following corollary to Theorem 5.1.

Corollary 5.1

$$e_1(0, \theta_{21}) > 1 \quad \text{if } \theta_{21} > \theta_m$$

and

$$e_1(0, \theta_{21}) \leq 1 \quad \text{if } 1 \leq \theta_{21} \leq \theta_m.$$

Several graphs are presented in Figure 5.1 to demonstrate the behavior of $e_1(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=0.5$.

We shall now consider the case that θ_{21} is a fixed but arbitrary number. Since $\sigma_1^2 \leq \sigma_2^2$, $\theta_{21} \geq 1$. Then we have the following lemma and theorem parallel to Lemma 4.3 and Theorem 4.2 respectively.

Lemma 5.3 For any given $\theta_{21} \geq 1$,

$$\lim_{\lambda \rightarrow \infty} e_1(\lambda, \theta_{21}) = 1^-.$$

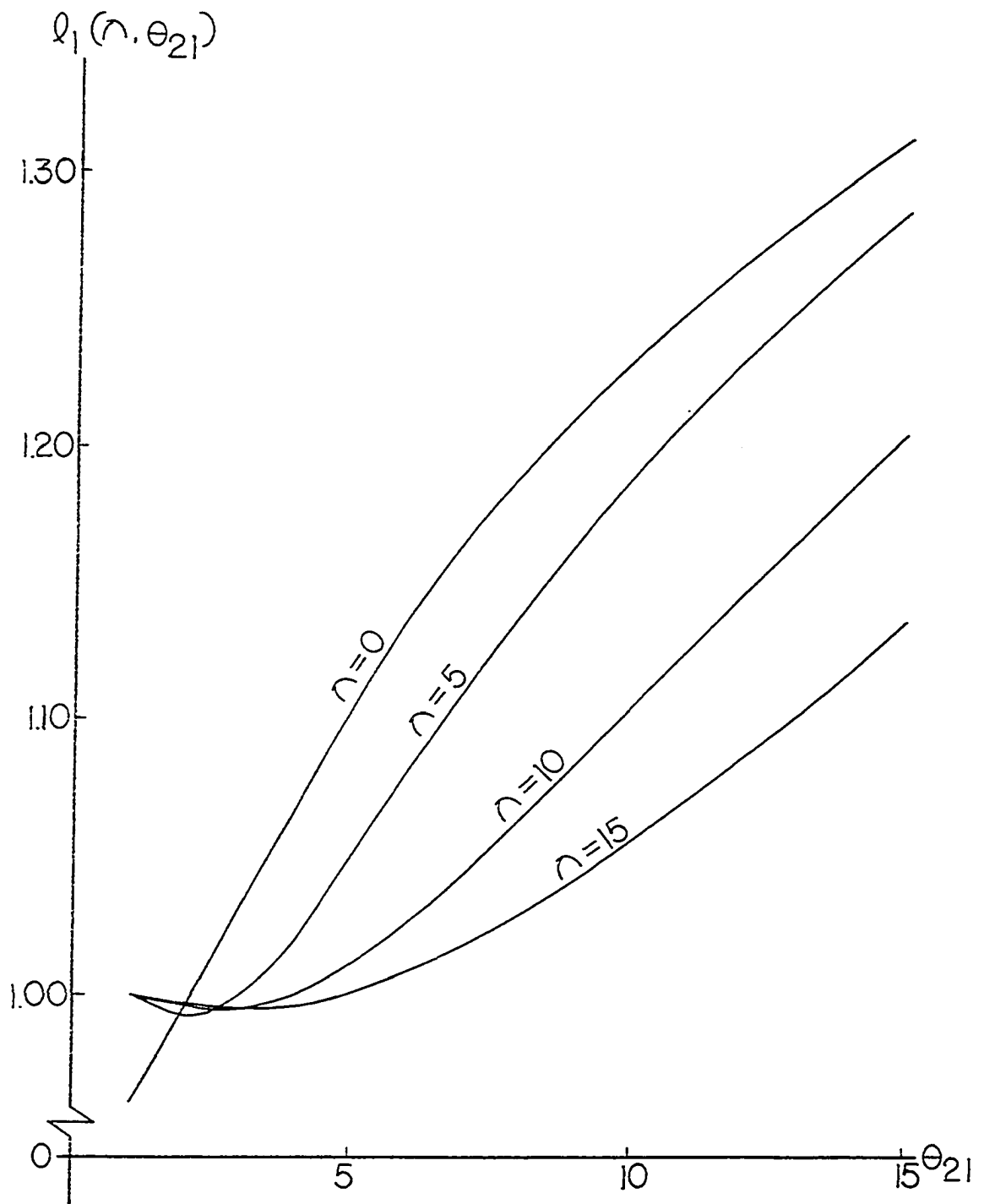


Figure 5.1. Graphs of $e_1(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

Theorem 5.2 Given $\theta_{21} \geq \theta_m$ where θ_m is defined in 4.1.10, $\exists \lambda_0$ such that

$$e_1(\lambda_0, \theta_{21}) = 1$$

and

$$e_1(\lambda, \theta_{21}) \geq 1 \quad \text{for each } \lambda \leq \lambda_0 .$$

Several graphs of $e_1(\lambda, \theta_{21})$ are presented in Figure 5.2.

Some further important properties of $e_1(\lambda, \theta_{21})$ will be given in the next two theorems.

Theorem 5.3 Let $\lambda_1 < \lambda_2$ be two values of λ .

$\exists \theta^* \geq 1$ such that

$$e_1(\lambda_1, \theta^*) = e_1(\lambda_2, \theta^*)$$

and

$$e_1(\lambda_1, \theta_{21}) > e_1(\lambda_2, \theta_{21}) \quad \text{for each } \theta_{21} > \theta^* .$$

Proof In Lemma 5.1, we have shown that $e_1(\lambda, 1)$ as well as $D_1(\lambda, 1)$ is an increasing function in λ when $\lambda \neq 1$. Therefore, with $\lambda_1 < \lambda_2$

$$e_1(\lambda_1, 1) < e_1(\lambda_2, 1)$$

and also

$$D_1(\lambda_1, 1) < D_1(\lambda_2, 1) .$$

For any given $\theta_{21} \geq 1$

$$D_1(\lambda_1, \theta_{21}) - D_1(\lambda_2, \theta_{21}) = (1 + \theta_{21}) d_0(\lambda_1, \lambda_2) - G \theta_{21}^{\frac{1}{2}} d_{-\frac{1}{2}}(\lambda_1, \lambda_2), \quad (5.1.3)$$

$$\text{where } d_i(\lambda_1, \lambda_2) = I_i\left(\frac{\theta_{21}}{\theta_{21} + \lambda_1}\right) - I_i\left(\frac{\theta_{21}}{\theta_{21} + \lambda_2}\right) . \quad (5.1.4)$$

Since we assume $\lambda_1 < \lambda_2$ then $d_i(\lambda_1, \lambda_2) > 0$ for any $i \leq \frac{1}{2}$ f. It can be verified that

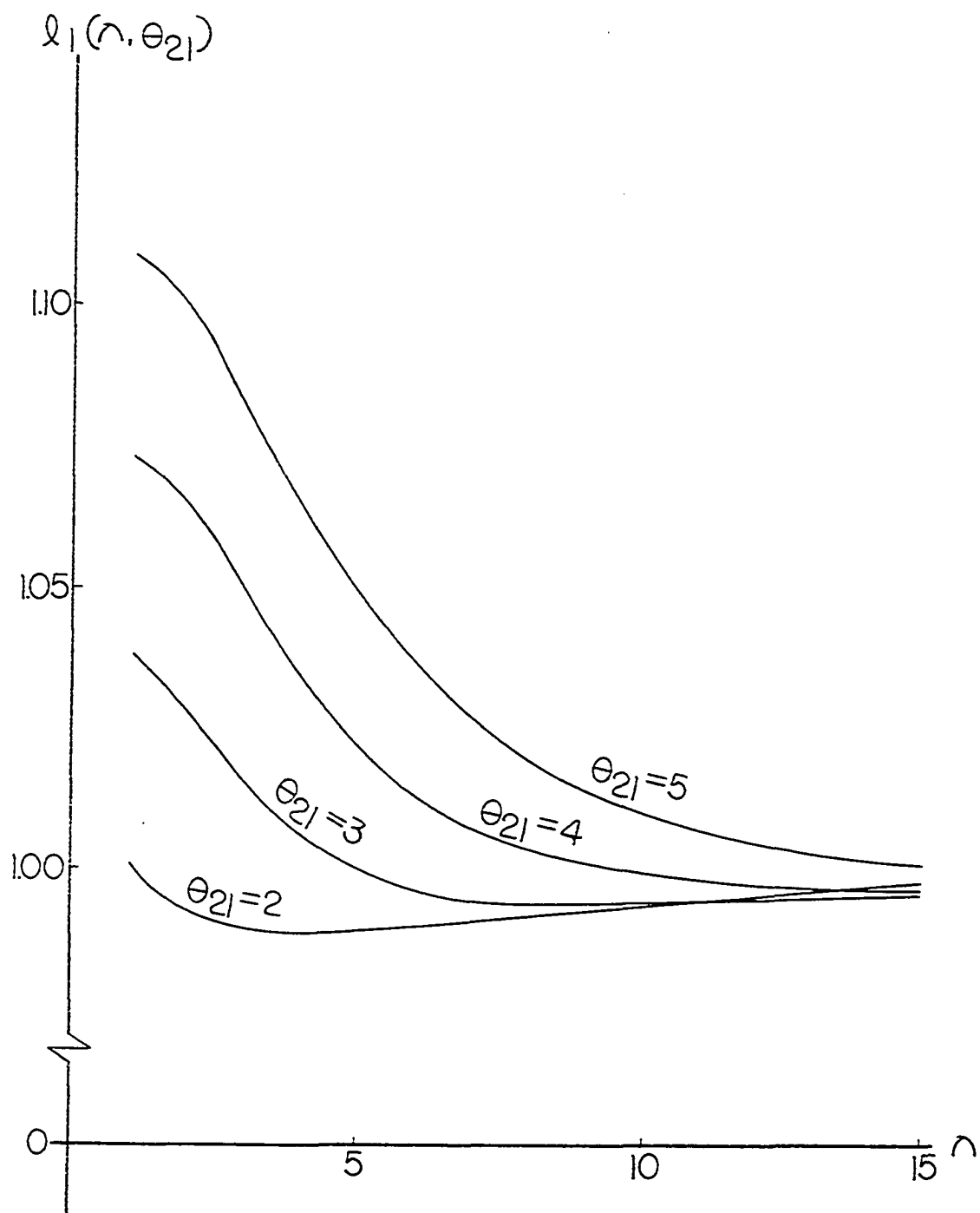


Figure 5.2. Graphs of $e_1(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

$$\lim_{\theta_{21} \rightarrow \infty} d_i(\lambda_1, \lambda_2) = 0^+. \quad (5.1.5)$$

Using 5.1.5, it can be seen that

$$\begin{aligned} \lim_{\theta_{21} \rightarrow \infty} \frac{1}{\theta_{21}} [D_1(\lambda_1, \theta_{21}) - D_2(\lambda_2, \theta_{21})] \\ = \lim_{\theta_{21} \rightarrow \infty} \frac{1}{\theta_{21}} (1 + \theta_{21}^{-G} \theta_{21}^{\frac{1}{2}}) 0^+ = 0^+. \end{aligned}$$

Therefore, $D_1(\lambda_1, \theta_{21}) - D_2(\lambda_2, \theta_{21})$ tends to a positive value as θ_{21} tends to infinity. Hence, there must be a $\theta^* \geq 1$ such that

$$D_1(\lambda_1, \theta^*) = D_1(\lambda_2, \theta^*)$$

and

$$D_1(\lambda_1, \theta_{21}) > D_1(\lambda_2, \theta_{21}) \quad \text{for each } \theta_{21} \geq \theta^*.$$

That, in turn, implies that

$$e_1(\lambda_1, \theta^*) = e_1(\lambda_2, \theta^*)$$

and

$$e_1(\lambda_1, \theta_{21}) > e_1(\lambda_2, \theta_{21}) \quad \text{for each } \theta_{21} \geq \theta^*.$$

The proof is completed.

The geometrical interpretation of Theorem 5.3 may be described as follows. For two distinct values, λ_1 and λ_2 of λ such that $\lambda_1 < \lambda_2$, $e_1(\lambda_1, \theta_{21})$ and $e_1(\lambda_2, \theta_{21})$ are represented by two curves. At $\theta_{21}=1$, the graph of $e_1(\lambda_2, \theta_{21})$ is above the graph of $e_1(\lambda_1, \theta_{21})$. As θ_{21} increases the two curves are getting closer. Then they intersect at a point at $\theta_{21}=\theta^*$. For each $\theta_{21} > \theta^*$, the graph of $e_1(\lambda_1, \theta_{21})$ will be on top of that of $e_1(\lambda_2, \theta_{21})$ and will stay on top. This phenomenon is indicated in Figure 5.1. When $\lambda_1=5$ and $\lambda_2=10$, θ^* is approximately equal to 2.5. Thus

$$e_1(5, \theta_{21}) < e_1(10, \theta_{21}) \quad \text{for each } \theta_{21} < 2.5$$

$$e_1(5, 2.5) = e_1(10, 2.5)$$

and

$$e_1(5, \theta_{21}) > e_1(10, \theta_{21}) \quad \text{for each } \theta_{21} > 2.5.$$

Theorem 5.4 Let e_0 be a real number such that $0 < e_0 < 1$. Then, $\exists \lambda_1^* \leq \lambda_2^*$ such that

$$e_1(\lambda, \theta_{21}) \geq e_0 \quad \text{for each } \lambda \leq \lambda_1^* \quad \text{or} \quad \lambda \geq \lambda_2^*.$$

Proof To prove this theorem, let $\theta_{21} \geq 1$ be a fixed number.

First, consider the case when $\min_{\lambda} e_1(\lambda, \theta_{21}) \geq e_0$. Then, $e_1(\lambda, \theta_{21}) \geq e_0$ for each λ . If we take $\lambda_1^* = \lambda_2^*$ to be any positive real number, then the theorem holds.

Now, consider the case when $\min_{\lambda} e_1(\lambda, \theta_{21}) < e_0$. Then, for some values of λ $e_1(\lambda, \theta_{21}) < e_0$. But $e_1(\lambda, \theta_{21})$ is decreasing when $1 < \lambda < \theta_{21}^2$. Also $\lim_{\lambda \rightarrow 1} e_1(\lambda, \theta_{21}) > 1$. It follows that $\exists \underline{\lambda}$ such that $e_1(\lambda, \theta_{21}) \geq e_0$ for each $\lambda < \underline{\lambda}$. Let

$$L_1 = \{ \underline{\lambda} : e_1(\lambda, \theta_{21}) \geq e_0 \quad \text{for each } \lambda < \underline{\lambda} \quad \text{and } \theta_{21} \geq 1 \}.$$

Clearly, $\inf_{\theta_{21} \geq 1} L_1$ is the required λ_1^* . On the other hand,

since $e_1(\lambda, \theta_{21})$ is increasing when $\lambda > \theta_{21}^2$ and $\lim_{\lambda \rightarrow \infty} e_1(\lambda, \theta_{21}) = 1^-$,

$\exists \bar{\lambda}$ such that $e_1(\lambda, \theta_{21}) \geq e_0$ for each $\lambda > \bar{\lambda}$. Let

$$L_2 = \{ \bar{\lambda} : e_1(\lambda, \theta_{21}) \geq e_0 \quad \text{for each } \lambda > \bar{\lambda} \quad \text{and } \theta_{21} \geq 1 \}.$$

Clearly, $\sup_{\theta_{21} \geq 1} L_2$ is the required λ_2^* .

Q.E.D.

From Theorem 5.4, we can see that given any $\theta_{21} \geq 1$ $\exists \lambda_1^* \leq \lambda_2^*$ such that for each $\lambda < \lambda_1^*$ or $\lambda > \lambda_2^*$ the efficiency of sometimes proportional allocation with respect to proportional allocation will never be lower than a pre-assigned value e_0 .

B. Efficiency with Respect to Modified

Neyman Allocation for $\sigma_1^2 \leq \sigma_2^2$

Now, we shall discuss the relative efficiency of sometimes proportional allocation with respect to modified Neyman allocation. The relative efficiency of sometimes proportional allocation with respect to modified Neyman allocation will be denoted by $e_2(\lambda, \theta_{21})$.

$$e_2(\lambda, \theta_{21}) = V(\bar{y}_w)_N / V(\bar{y}_w)_S = 1 / \left[1 - \frac{w_1 w_2 D_2(\lambda, \theta_{21})}{w_1^2 + w_2^2 \theta_{21} + w_1 w_2 G \theta_{21}^{\frac{1}{2}}} \right], \quad (5.2.1)$$

where $D_2(\lambda, \theta_{21})$ is defined in 4.2.1.

Clearly, if $e_2(\lambda, \theta_{21}) > 1$, sometimes proportional allocation is more efficient than modified Neyman allocation. From 5.2.1, we see that the behavior of $e_2(\lambda, \theta_{21})$ is closely related to that of $D_2(\lambda, \theta_{21})$. In particular, when θ_{21} is fixed but arbitrary, the behavior of $e_2(\lambda, \theta_{21})$ will be the same as that of $D_2(\lambda, \theta_{21})$.

Since $e_2(\lambda, \theta_{21})$ and $D_2(\lambda, \theta_{21})$ are closely related, some of the results of $e_2(\lambda, \theta_{21})$ parallel to those of $D_2(\lambda, \theta_{21})$ can be obtained quite easily.

Lemma 5.4 For any $\lambda \geq 0$, $\lim_{\theta_{21} \rightarrow 1} e_2(\lambda, \theta_{21}) \geq 1$.

Furthermore, $e_2(\lambda, 1)$ is an increasing function in λ except when $\lambda=1$.

Next, for a fixed but arbitrary $\lambda \geq 0$, we have the following theorem, which is a direct consequence of Theorem 4.3.

Theorem 5.5 For any given $\lambda \geq 0$, $\exists \theta_0 \geq 1$ such that

$$e_2(\lambda, \theta_0) = 1$$

and

$$e_2(\lambda, \theta_{21}) \geq 1 \quad \text{for each } \theta_{21} \leq \theta_0.$$

The corresponding results for θ_{21} arbitrary but fixed are given below.

Lemma 5.5 For any given $\theta_{21} \geq 1$
 $\lim_{\lambda \rightarrow 0} e_2(\lambda, \theta_{21}) = 1^+.$

Theorem 5.6 For any given $\theta_{21} > \theta_m$ where θ_m is defined in 4.1.10, $\exists \lambda_0$ such that

$$e_2(\lambda_0, \theta_{21}) = 1$$

and

$$e_2(\lambda, \theta_{21}) \geq 1 \quad \text{for each } \lambda \leq \lambda_0.$$

Several graphs of $e_2(\lambda, \theta_{21})$ are presented in Figures 5.3 and 5.4.

The next two theorems are analogous to Theorems 5.3 and 5.4.

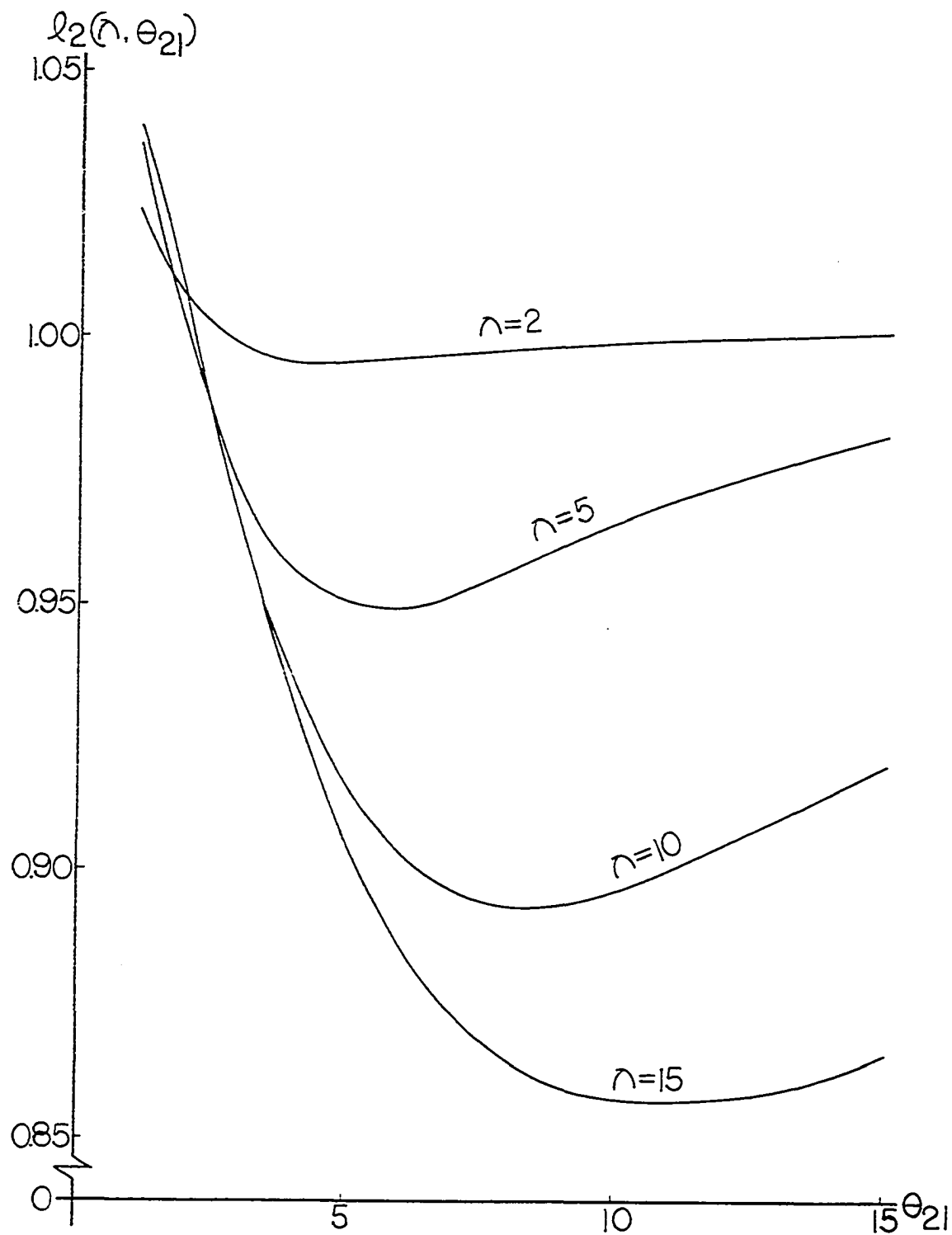


Figure 5.3. Graphs of $e_2(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

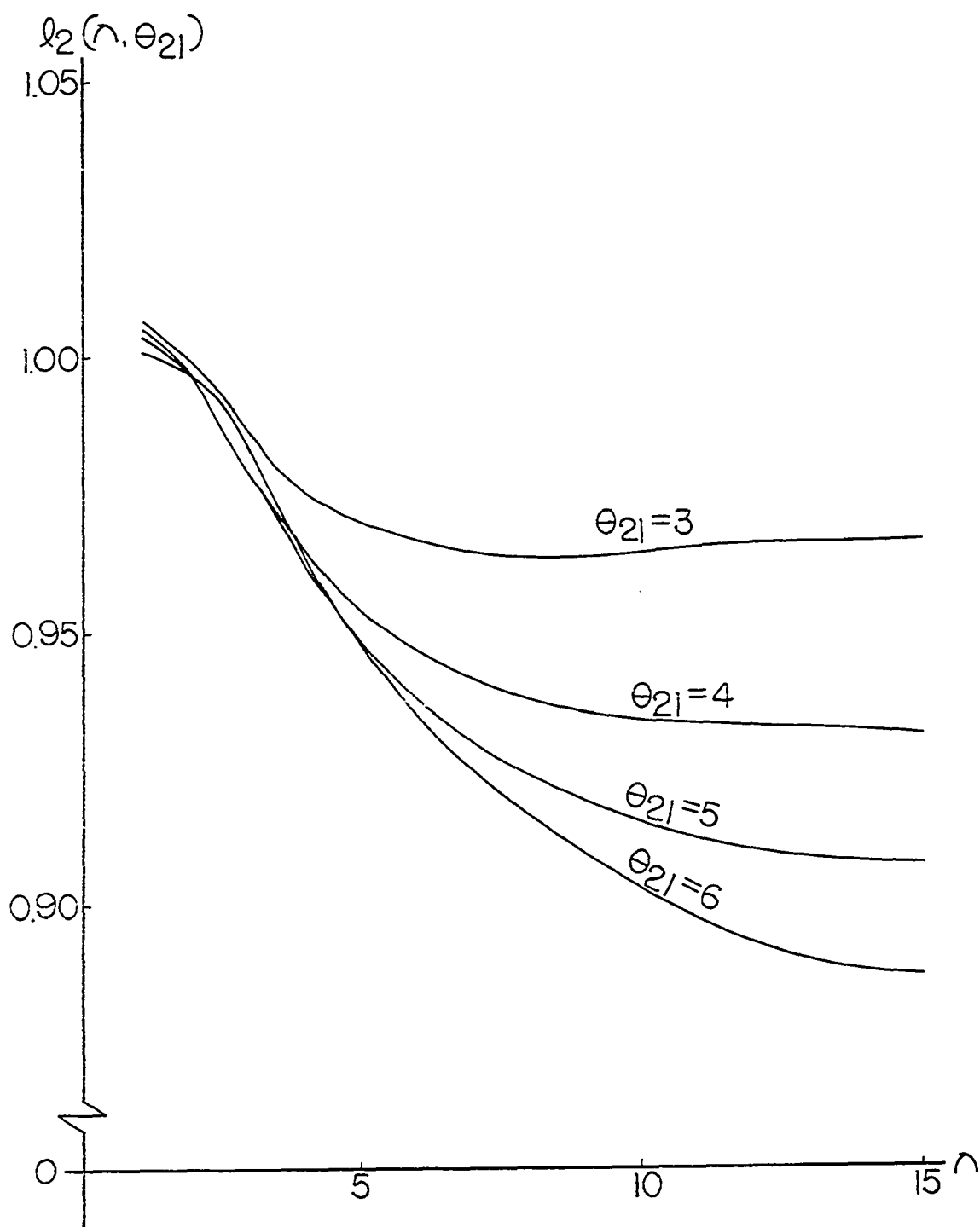


Figure 5.4. Graphs of $e_2(\lambda, \theta_{21})$ for $m=8$ and $w_1=w_2=\frac{1}{2}$

Theorem 5.7 Let $\lambda_1 < \lambda_2$ be two values of λ
 if $\theta^* \geq 1$ such that

$$e_2(\lambda_1, \theta^*) = e_2(\lambda_2, \theta^*)$$

and

$$e_2(\lambda_1, \theta_{21}) > e_2(\lambda_2, \theta_{21}) \quad \text{for each } \theta_{21} > \theta^*.$$

Theorem 5.8 Let e_0 be a real number such that
 $0 < e_0 < 1$. Then $\exists \lambda_1^* \leq \lambda_2^*$ such that

$$e_2(\lambda, \theta_{21}) \geq e_0 \quad \text{for each } \lambda \leq \lambda_1^* \text{ or } \lambda \geq \lambda_2^*.$$

For the purpose of illustration, consider the problem of sampling households in a town in order to estimate the average amount of assets per household that are readily convertible into cash. The households are stratified into a high-rent and a low-rent stratum. The variance σ_2^2 in the high-rent stratum is considerably higher than the variance σ_1^2 in the low-rent stratum. On the basis of preliminary evidence, it is guessed that $\theta_{21} \leq 9$. It is known that

$$N = 24,000 \quad w_1 = 5/6 \quad \text{and} \quad w_2 = 1/6$$

N_1 and N_2 are sufficiently large, so that finite correction factors can be ignored. Further, let $f=7$ and $\lambda=2$. The table below gives the relative efficiency of sometimes proportional allocation with respect to proportional allocation as also with respect to modified Neyman allocation for different values of θ_{21} .

Table 5.1. Relative efficiency of sometimes proportional allocation

| With respect to | $\theta_{21}=1$ | $\theta_{21}=3$ | $\theta_{21}=5$ | $\theta_{21}=7$ | $\theta_{21}=9$ |
|----------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Proportional Allocation | 0.99 | 1.02 | 1.10 | 1.18 | 1.26 |
| Modified Neyman Allocation | 1.014 | 0.998 | 0.995 | 0.997 | 0.998 |

It is seen that for appropriate choice of the level of significance as determined by λ (in this case $\lambda=2$), sometimes proportional allocation is almost as efficient as modified Neyman allocation. It is also seen that sometimes proportional allocation is almost as efficient as proportional allocation for values of θ_{21} close to 1 while it is considerably more efficient than proportional allocation for values of θ_{21} closer to 9.

C. Efficiency with Respect to Proportional Allocation for $\sigma_1^2 \neq \sigma_2^2$

Consider the relative efficiency of sometimes proportional allocation with respect to proportional allocation,

$$e_1^*(\lambda, \theta_{21}) = V(\bar{y}_w)_P / V(\bar{y}_w)_S = 1 / \left[1 - \frac{w_1 w_2}{w_1 + w_2 \theta_{21}} D_1^*(\lambda, \theta_{21}) \right], \quad (5.3.1)$$

where $D_1^*(\lambda, \theta_{21})$ is defined in 4.3.1.

Similar to what we have mentioned earlier, if $e_1^*(\lambda, \theta_{21}) > 1$, sometimes proportional allocation is more efficient than proportional allocation. From 5.3.1, we also can see that $e_1^*(\lambda, \theta_{21})$ will behave in the same manner as $D_1^*(\lambda, \theta_{21})$. Furthermore, some results of $e_1^*(\lambda, \theta_{21})$ parallel to those of $D_1^*(\lambda, \theta_{21})$ which have been presented in section IV C will be given below.

First let us consider the case that λ is a fixed but arbitrary number. We have the following theorem for $e_1^*(\lambda, \theta_{21})$.

Theorem 5.9 For any given $\lambda \geq 1$ satisfying 4.3.11, $\exists \theta_0^{(1)}$ in $(0, 1)$ and $\theta_0^{(2)} > 1$ such that

$$e_1^*(\lambda, \theta_0^{(1)}) = e_1^*(\lambda, \theta_0^{(2)}) = 1$$

and

$$e_1^*(\lambda, \theta_{21}) > 1 \quad \forall \quad \theta_{21} < \theta_0^{(1)} \quad \text{or} \quad \theta_{21} > \theta_0^{(2)}.$$

Next we shall consider the case that θ_{21} is a fixed but arbitrary number. $e_1^*(\lambda, \theta_{21})$ and $D_1^*(\lambda, \theta_{21})$ will behave exactly the same. Then the behavior of $e_1^*(\lambda, \theta_{21})$, as a function of λ , will be given in the following theorem.

Theorem 5.10 For any given θ_{21} in S' , where S is defined in 4.3.16, $\exists \lambda_0$ such that

$$e_1^*(\lambda_0, \theta_{21}) = 1$$

and

$$e_1^*(\lambda, \theta_{21}) \geq 1 \quad \forall \quad \lambda \leq \lambda_0.$$

D. Efficiency with Respect to Modified
Neyman Allocation for $\sigma_1^2 \neq \sigma_2^2$

Now we shall discuss the relative efficiency of sometimes proportional allocation with respect to modified Neyman allocation. The relative efficiency of sometimes proportional allocation with respect to modified Neyman allocation will be denoted by $e_2^*(\lambda, \theta_{21})$.

$$e_2^*(\lambda, \theta_{21}) = V(\bar{y}_w)_N / V(\bar{y}_w)_S = 1 / \left[1 - \frac{w_1 w_2 D_2^*(\lambda, \theta_{21})}{w_1^2 + w_2^2 \theta_{21} + w_1 w_2 G \theta_{21}^{\frac{1}{2}}} \right], \quad (5.4.1)$$

where $D_2^*(\lambda, \theta_{21})$ is defined in 4.4.1.

Again, if $e_2^*(\lambda, \theta_{21}) > 1$, sometimes proportional allocation is more efficient than modified Neyman allocation. From 5.4.1, we also can see that $e_2^*(\lambda, \theta_{21})$ will behave in the same manner as $D_2^*(\lambda, \theta_{21})$. Furthermore, some results of $e_2^*(\lambda, \theta_{21})$ parallel to those of $D_2^*(\lambda, \theta_{21})$ which have been presented in section IV D will be given below.

First let us consider the case that λ is a fixed but arbitrary number. We have the following theorem for $e_2^*(\lambda, \theta_{21})$.

Theorem 5.11 For any given $\lambda \geq 1$, $\exists \theta_0^{(1)}$ in $(0, 1)$ and $\theta_0^{(2)} > 1$ such that

$$e_2^*(\lambda, \theta_0^{(1)}) = e_2^*(\lambda, \theta_0^{(2)}) = 1$$

and

$$e_2^*(\lambda, \theta_{21}) \geq 1 \quad \forall \quad \theta_{21} \quad \text{in} \quad [\theta_0^{(1)}, \theta_0^{(2)}].$$

Next we shall consider the case that θ_{21} is fixed but arbitrary number. $e_2^*(\lambda, \theta_{21})$ and $D_2^*(\lambda, \theta_{21})$ will behave exactly the same. Then the behavior of $e_2^*(\lambda, \theta_{21})$, as a function of λ , will be given in the following theorem.

Theorem 5.12 For any given θ_{21} in S , where S is defined in 4.3.6, $\exists \lambda_0$ such that

$$e_2^*(\lambda_0, \theta_{21}) = 1$$

and

$$e_2^*(\lambda, \theta_{21}) \geq 1 \quad \forall \quad \lambda \geq \lambda_0.$$

VI. BIBLIOGRAPHY

Bancroft, T. A. 1964. Inference for incompletely specified models involving use of preliminary test(s) of significance. *Biometrics* 20: 427-442.

Carrillo, Alfonso Fausto. 1969. Estimation of variance after preliminary tests of significance. Unpublished Ph.D. thesis. Ames, Iowa, Library, Iowa State University of Science and Technology.

Draper, N. R. and Guttman, I. 1968a. Some Bayesian stratified two-phase sampling results. *Biometrika* 55: 131-139.

_____ and _____ 1968b. Some Bayesian stratified two-phase sampling results: k characteristics. *Biometrika* 55: 587-589.

Ericson, W. A. 1965. Optimum stratified sampling using prior information. *Journal of American Statistical Association* 60: 750-771.

_____ 1968. Optimal allocation in stratified and multi-stage samples using prior information. *Journal of American Statistical Association* 63: 964-985.

Evans, W. D. 1951. On stratification and optimum allocations. *Journal of American Statistical Association* 46: 95-104.

Grosh, D. L. 1969. Bayes one-and-two stage stratified sampling schemes for finite populations with beta-binomial prior distributions. Unpublished Ph.D. thesis. Manhattan, Kansas, Library, Kansas State University.

Kitagawa, T. 1963. Estimation after preliminary tests of significance. University of California, Berkeley, California, Publications on Statistics 3: 147-186.

Ruhl, D. J. and Sedransk, J. H. 1967. Combining information from two or more sample surveys. Unpublished paper presented at the Central Regional Meeting of I.M.S., Ohio State Univ., Columbus, Ohio. Ames, Iowa, Statistical Laboratory, Iowa State University of Science and Technology.

Solomon, H. and Zacks, S. 1970. Optimal design of sampling from finite populations: A critical review and indication of new research areas. *Journal of American Statistical Association* 65: 653-677.

Sukhatme, P. V. 1935. Contribution to the theory of the representative method. Journal of Royal Statistical Society Supplement 2: 253-268.

Zacks, S. 1970. Bayesian Design of single and double stratified sampling for estimating proportion in finite population. Technometrics 12: 119-130.

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